Obliquely Reflected Brownian Motion in Nonsmooth Domains with Fractional and Subfractional Noise: A Transportation Systems Framework

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Abstract

Transportation is the art of moving people and goods across space and time, but it is also a reflection of the political and economic forces. Sometimes, these forces are smooth and predictable, like a well-designed highway. Other times, they are rough and chaotic, like a bumpy road or a turbulent crisis. In either case, we propose to model the behavior of transportation systems using obliquely reflected Brownian motion in nonsmooth domains with fractional and subfractional noise, which captures the randomness, the constraints, and the complexity of the real world. We apply and extend important results from the probability and statistics literature on obliquely reflected Brownian motion in nonsmooth planar domains to account for fractional and subfractional noise. We consider understudied realistic scenarios that involve complex road network topologies, different traffic conditions at the boundary, and memory effects in the drivers' behavior. We use a multiple connected domain to capture the presence of holes or islands in the road network, a switching or regime-switching model to account for absorption or tangential motion at the boundary, and a fractional Brownian motion or a fractional diffusion process to incorporate long-range dependence or memory in the process. We construct and analyze the generalized obliquely reflected Brownian motion in these settings and explore its potential applications to queuing theory and traffic flow optimization, especially in settings such as transportation in large urban centers.

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1 Introduction

As with other queuing theory applications, transportation is the art of moving people and goods across space and time, but it is also a reflection of the political and economic forces that shape our world. Sometimes, these forces are rather smooth and predictable, like on a straightforward highway. Other times, they are rough and chaotic, like a more complex road system. In either case, one policy challenge is to model the behavior of transportation systems that *exactly* captures the randomness, the constraints, and the complexity of the real world.

The goal of this paper is to introduce a methodology for doing so, using obliquely reflected Brownian motion in nonsmooth domains with fractional and subfractional noise. Obliquely reflected Brownian motion (ORBM) is a stochastic process that describes the random motion that is confined to a bounded domain and reflects obliquely at the boundary according to a given reflection vector field. This process can be used to model various physical systems, such as diffusion of molecules, heat transfer, or fluid flow. ORBM was first introduced by Skorokhod (1961) and has been extensively studied, Williams (1987), Lions and Sznitman (1984), Burdzy, Chen and Sylvester (2004), and many others[1]

One of the main challenges in the theory of ORBM is to construct and characterize the process in general domains and with general reflection vector fields. In particular, the case of nonsmooth domains and nonsmooth reflection vector fields poses significant difficulties due to the lack of regularity and uniqueness of the solutions of the associated partial differential equations. In a seminal result in probability theory and statistics, Burdzy et al. (2017) overcame these difficulties by using conformal mappings and excursion theory, and provided a general construction and characterization of ORBM in any bounded simply connected planar domain, including nonsmooth domains, with any continuous and nonvanishing reflection vector field on the boundary. They also obtained some important properties of ORBM, such as the stationary distribution, the rate of rotation, and the limit behavior.

In this paper, we apply and extend the results of Burdzy et al. (2017) to model unprecedented levels of realism, such as the movement of cars in a traffic jam. We thus consider more realistic

[1] See Ramamnan (2006) for basic applied probabilistic models in queuing theory; Holyst et al. (2000) for biological applications; Dupuis and Ishii (2008) for multidimensional domain studies as well as Burdzy et al. (2017) and the references therein.

scenarios that involve complex road network topologies, different traffic conditions at the boundary, and memory effects in the drivers' behavior. We use a multiple connected domain to capture the presence of holes or islands in the road network, a switching or regime-switching model to account for absorption or tangential motion at the boundary, and a fractional Brownian motion or a fractional diffusion process to incorporate long-range dependence or memory in the process. We construct and analyze the generalized obliquely reflected Brownian motion in these settings and explore its potential applications to queuing theory and traffic flow optimization.

The contributions are as follows. One of the main advantages of our model is that it allows for holes or islands in the domain, which are regions that are inaccessible or forbidden for the process. This feature is realistic for some road networks that have complex topologies, such as bridges, tunnels, overpasses, underpasses, roundabouts, or intersections. For example, consider a road network that consists of two parallel roads connected by a bridge. The bridge can be modeled as a hole or an island in the domain, since the process cannot enter or exit the bridge except at its endpoints. To accommodate holes or islands in the domain, we use a multiple connected domain and extend the definition and construction of the obliquely reflected Brownian motion accordingly. We show that our model preserves the properties of existence, uniqueness, stationarity, rotation, and limit behavior of the process in multiple connected domains.

Another advantage of our model is that it allows for absorption or tangential motion at the boundary, which are types of boundary behavior that differ from oblique reflection. Absorption means that the process stops or terminates when it hits the boundary, while tangential motion means that the process slides along the boundary without changing its direction. These types of boundary behavior are realistic for some traffic situations that involve stopping, turning, or changing lanes at the boundary. For example, consider a traffic light at an intersection. The traffic light can be modeled as an absorbing boundary, since the process stops when it reaches the traffic light. Alternatively, consider a curve or a bend in a road. The curve or bend can be modeled as a tangential boundary, since the process slides along the curve or bend without changing its direction. To accommodate absorption or tangential motion at the boundary, we use a switching or regime-switching model, which allows for different types of boundary behavior depending on some random or deterministic factors. We show that our model preserves the properties of existence, uniqueness, stationarity, rotation, and limit behavior of the process in switching or regime-switching models.

A third advantage of our model is that it allows for fractional and subfractional noise in the process, which are types of noise that differ from standard Brownian noise. Fractional noise means that the process depends on its current state and also on its past history, while subfractional noise means that the process depends on its current state and also on its future history. These types of noise are realistic for some traffic situations that involve memory effects, such as anticipation, adaptation, or learning by the drivers. For example, consider a traffic jam on a highway. The traffic jam can be modeled as a fractional noise, since the process depends on its current speed and also on its past speed. Alternatively, consider a traffic signal ahead on a road. The traffic signal can be modeled as a subfractional noise, since the process depends on its current speed and also on its future speed. To accommodate fractional and subfractional noise in the process, we use a fractional Brownian motion or a fractional diffusion process, which are two generalizations of the Brownian motion that have long-range dependence or memory. We show that our model preserves the properties of existence, uniqueness, stationarity, rotation, and limit behavior of the process in fractional and subfractional models.

In summary, our model of obliquely reflected Brownian motion in nonsmooth domains with fractional and subfractional noise is more realistic and flexible than the approach of Burdzy et al. (2017) for analyzing traffic and other queuing systems. Our model can capture more complex features of these systems, such as holes or islands in the domain, absorption or tangential motion at the boundary, and fractional and subfractional noise in the process. We believe that our model can provide more accurate and useful insights into these systems and their dynamics.

The rest of the paper is organized as follows. In Section 2, we review some preliminaries on ORBM, conformal mappings, excursion theory, and fractional processes. In Section 3, we present our main results on the construction and characterization of ORBM in multiple connected domains with switching or regime-switching reflection vector fields. In Section 4, we discuss some properties and applications of ORBM with fractional components. In Section 5, we conclude with some remarks and open problems.

2 Preliminaries

In this section, we review some basic concepts and results on ORBM, conformal mappings, excursion theory, and fractional processes that will be used throughout the paper.

2.1 Obliquely reflected Brownian motion

Let D be a bounded domain in $\mathbb{R}^2$ with boundary ∂D. Let ν be the unit outward normal vector on ∂D. Let b be a continuous vector field on ∂D such that $b \cdot \nu \neq 0$ for all $x \in \partial D$. The vector field b determines the angle of oblique reflection at the boundary. Let $W = (W_1, W_2)$ be a two-dimensional standard Brownian motion starting from a point $x_0 \in D$. An obliquely reflected Brownian motion (ORBM) in D with reflection vector field b is a continuous process $X = (X_1, X_2)$ that satisfies the following stochastic differential equation (SDE):

$$dX_t = dW_t + \lambda_t b(X_t)dt, \quad X_0 = x_0,$$

where λ_t is a nondecreasing process that represents the local time of X on ∂D. The process λ_t is determined by the Skorokhod reflection condition:

$$X_t - x_0 - W_t \in \overline{D} \quad \text{for all } t \geq 0.$$

The process X can be interpreted as a Brownian motion that is confined to the domain D and reflects obliquely at the boundary according to the vector field b. The process λ_t measures the amount of time that X spends on ∂D. The process X is Markovian and has a unique strong solution under some regularity conditions on D and b, such as Lipschitz continuity or uniform ellipticity.

One of the main results of Burdzy et al. (2017) is that ORBM can be constructed and characterized in any bounded simply connected planar domain, including nonsmooth domains, with any continuous and nonvanishing reflection vector field on the boundary. They also showed that ORBM has a stationary distribution, which is given by an integrable positive harmonic function in D, and a rate of rotation, which is given by a real number that represents the asymptotic angular speed of X around a reference point in D. Moreover, they proved that ORBM converges to a point mass at the center of mass of its stationary distribution as the domain shrinks to a point.

2.2 Conformal mappings

A conformal mapping is a function $f : D \to D'$ between two domains in $\mathbb{C}$ that preserves angles locally. Equivalently, a conformal mapping is an analytic function that has nonzero derivative everywhere in its domain. Conformal mappings are useful tools for studying planar domains, as they can transform complex or irregular domains into simpler or more regular ones, while preserving some geometric or analytic properties.

One of the main results of complex analysis is the Riemann mapping theorem, which states that any simply connected domain in $\mathbb{C}$, other than $\mathbb{C}$ itself, can be conformally mapped onto the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Moreover, such a conformal mapping is unique up to a rotation. The Riemann mapping theorem can be extended to multiple connected domains by using the concept of prime ends, which are equivalence classes of curves that approach the boundary of the domain. The Carathéodory theorem states that any finitely connected domain in $\mathbb{C}$ can be conformally mapped onto a circular domain, which is a domain whose boundary consists of finitely many disjoint circles.

Burdzy et al. (2017) used conformal mappings to construct and characterize ORBM in simply connected domains. They showed that if $f : D \to D'$ is a conformal mapping between two simply connected domains, and if X is an ORBM in D with reflection vector field b, then $Y = f(X)$ is an ORBM in D' with reflection vector field $c = f'(X)b(X)/|f'(X)|$. They also showed how to compute the stationary distribution and the rate of rotation of ORBM in terms of the conformal mapping and its derivative.

2.3 Excursion theory

An excursion of a stochastic process X is a segment of the sample path of X that starts and ends at a fixed state, usually zero. Excursion theory is a branch of probability theory that studies the properties and distributions of excursions of stochastic processes, such as Brownian motion, Lévy processes, or Markov processes. Excursion theory can be used to analyze various phenomena that involve crossing, hitting, or exiting certain states or regions by stochastic processes.

One of the main results of excursion theory is the Itô excursion theory, which provides a general framework for constructing and characterizing excursions of Markov processes from a measurable subset of the state space. The Itô excursion theory states that there exists a unique excursion measure $\mathbb{N}$ on the space of excursions, such that for any Markov process X and any measurable subset A of the state space, the number of excursions of X from A during a time interval $[0, T]$ has a Poisson distribution with mean $\int_0^T \mathbb{N}(A, X_s)ds$, where $\mathbb{N}(A, x)$ is the intensity of the excursion measure at state x and subset A. Moreover, the excursion measure $\mathbb{N}$ is related to the transition function and the potential measure of the Markov process X.

Burdzy et al. (2017) used excursion theory to construct and characterize ORBM in nonsmooth domains. They showed that if D is a simply connected domain with a nonsmooth boundary point x_0, and if b is a continuous and nonvanishing reflection vector field on ∂D, then there exists a unique ORBM in D with reflection vector field b that starts and ends at x_0. They also showed how to compute the distribution and the expectation of this excursion in terms of the conformal mapping, the reflection vector field, and the potential measure.

2.4 Fractional processes

A fractional process is a stochastic process that has some form of long-range dependence or memory. Long-range dependence or memory means that the autocorrelation or dependence function of the process decays slowly or remains positive as the time lag increases. Fractional processes can be used to model various phenomena that exhibit long-range dependence or memory, such as network traffic, hydrology, finance, or biology.

One of the most well-known fractional processes is the fractional Brownian motion (fBm), which is a generalization of the Brownian motion that has a self-similar and stationary increment structure with a Hurst parameter $H \in (0, 1)$. The Hurst parameter H determines the degree of long-range dependence or memory of the fBm. When $H = 1/2$, the fBm reduces to the standard Brownian motion, which has no long-range dependence or memory. When $H < 1/2$, the fBm has negative long-range dependence or memory, which means that its increments tend to alternate in sign. When $H > 1/2$, the fBm has positive long-range dependence or memory, which means that its increments tend to have the same sign.

Another important fractional process is the fractional diffusion process (fDp), which is a generalization of the diffusion process that has a fractional time derivative in its governing equation. The fractional time derivative is defined by using an integral operator with a power-law kernel, which captures the history or memory effects of the process. The order of the fractional time derivative $\alpha \in (0, 1)$ determines the degree of long-range dependence or memory of the fDp. When $\alpha = 1$, the fDp reduces to the standard diffusion process, which has no long-range dependence or memory. When $\alpha < 1$, the fDp has positive long-range dependence or memory, which means that its evolution depends on its entire past history.

3 ORBM in multiple connected domains with switching or regime-switching reflection vector fields

In this section, we present our main results on the construction and characterization of ORBM in multiple connected domains with switching or regime-switching reflection vector fields. We first introduce the setting and the notation, and then state and prove our main theorems.

3.1 Setting and notation

Let D be a bounded multiple connected domain in $\mathbb{R}^2$ with boundary ∂D. We assume that ∂D consists of $n \geq 2$ disjoint simple closed curves $\Gamma_1, \ldots, \Gamma_n$, which we call the components of ∂D. We assume that each component Γ_i is oriented counterclockwise and has a positive Jordan measure. We denote by D_i the bounded domain enclosed by Γ_i, and by D_0 the unbounded domain exterior to all Γ_i. We have $D = \overline{D_0} \setminus \bigcup_{i=1}^n \overline{D_i}$.

Let b be a continuous vector field on ∂D such that $b \cdot \nu \neq 0$ for all $x \in \partial D$, where ν is the unit outward normal vector on ∂D. The vector field b determines the angle of oblique reflection at the boundary. We assume that b is not constant on each component Γ_i, but may vary from one component to another. We also assume that b may switch or change its value according to some random or deterministic factors, such as the state of the process, the time, or an external signal. We call such a vector field a switching or regime-switching reflection vector field.

Let $W = (W_1, W_2)$ be a two-dimensional standard Brownian motion starting from a point $x_0 \in D$. We are interested in constructing and characterizing an obliquely reflected Brownian motion

(ORBM) in D with reflection vector field b. That is, a continuous process $X = (X_1, X_2)$ that satisfies the following stochastic differential equation (SDE):

$$dX_t = dW_t + \lambda_t b(X_t)dt, \quad X_0 = x_0,$$

where λ_t is a nondecreasing process that represents the local time of X on ∂D, and satisfies the Skorokhod reflection condition:

$$X_t - x_0 - W_t \in \overline{D} \quad \text{for all } t \geq 0.$$

We will use the following notation throughout this section:

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk in the complex plane.
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in the complex plane.
- For any domain $G \subset \mathbb{C}$, we denote by $\mathcal{H}(G)$ the space of harmonic functions on G, and by $\mathcal{O}(G)$ the space of analytic functions on G.
- For any function $f : G \to \mathbb{C}$, we denote by f' and f'' its first and second complex derivatives, respectively.
- For any curve $\gamma : [a, b] \to \mathbb{C}$, we denote by $\ell(\gamma)$ its arc length, and by $\kappa(\gamma)$ its signed curvature, defined as $\kappa(\gamma) = (\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')/|\gamma'|^3$, where $\gamma = (\gamma_1, \gamma_2)$.
- For any two curves $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$, we denote by $\langle \gamma_1, \gamma_2 \rangle$ their algebraic intersection number, defined as the sum of the signs of the cross products of their tangent vectors at their intersection points.
- For any Jordan curve Γ, we denote by $n(\Gamma)$ its winding number around zero, defined as $\frac{1}{2\pi i} \int_\Gamma dz/z$.

3.2 Main results

Our main results are the following two theorems, which provide a construction and a characterization of ORBM in multiple connected domains with switching or regime-switching reflection vector fields.

Theorem 3.1 (Construction)

Let D be a bounded multiple connected domain in $\mathbb{R}^2$ with boundary ∂D consisting of $n \geq 2$ disjoint simple closed curves $\Gamma_1, \ldots, \Gamma_n$. Let b be a continuous and nonvanishing switching or regime-switching reflection vector field on ∂D. Let W be a two-dimensional standard Brownian motion starting from a point $x_0 \in D$. Then there exists a unique strong solution X to the SDE

$$dX_t = dW_t + \lambda_t b(X_t)dt, \quad X_0 = x_0,$$

where λ_t is a nondecreasing process that satisfies the Skorokhod reflection condition

$$X_t - x_0 - W_t \in \overline{D} \quad \text{for all } t \geq 0.$$

The process X is an ORBM in D with reflection vector field b. Moreover, the process X can be constructed as follows:

- Let $f : D \to \mathbb{D}$ be a conformal mapping from D to the unit disk $\mathbb{D}$, such that $f(x_0) = 0$. Such a mapping exists and is unique by the Carathéodory theorem.
- Let $\tilde{\Gamma}_i = f(\Gamma_i)$ for $i = 1, \ldots, n$. Then $\tilde{\Gamma}_i$ are disjoint simple closed curves in $\mathbb{C}$ that enclose the origin. Let $\tilde{D}_i$ be the bounded domain enclosed by $\tilde{\Gamma}_i$, and let $\tilde{D}_0$ be the unbounded domain exterior to all $\tilde{\Gamma}_i$. We have $\mathbb{D} = \overline{\tilde{D}_0} \setminus \bigcup_{i=1}^n \overline{\tilde{D}_i}$.
- Let $\tilde{b}$ be a continuous and nonvanishing switching or regime-switching reflection vector field on $\partial\mathbb{D}$, defined by $\tilde{b}(z) = f'(f^{-1}(z))b(f^{-1}(z))/|f'(f^{-1}(z))|$ for $z \in \partial\mathbb{D}$. Then $\tilde{b}$ determines the same angle of oblique reflection as b on each component of $\partial\mathbb{D}$.
- Let $\tilde{W} = f(W)$ be a two-dimensional standard Brownian motion on $\mathbb{D}$ starting from the origin. Then there exists a unique strong solution $\tilde{X}$ to the SDE

  $$d\tilde{X}_t = d\tilde{W}_t + \tilde{\lambda}_t \tilde{b}(\tilde{X}_t)dt, \quad \tilde{X}_0 = 0,$$

  where $\tilde{\lambda}_t$ is a nondecreasing process that satisfies the Skorokhod reflection condition

$$\tilde{X}_t - \tilde{W}_t \in \overline{\mathbb{D}} \quad \text{for all } t \geq 0.$$

The process $\tilde{X}$ is an ORBM in $\mathbb{D}$ with reflection vector field $\tilde{b}$, which exists by the result of Burdzy et al. (2017).

- Let $X = f^{-1}(\tilde{X})$. Then X is an ORBM in D with reflection vector field b, which is the desired process.

Theorem 3.2 (Characterization)

Let D, b, W, and X be as in Theorem 3.1. Let f, $\tilde{\Gamma}_i$, $\tilde{D}_i$, $\tilde{b}$, $\tilde{W}$, and $\tilde{X}$ be as in the construction of Theorem 3.1. Then the following statements hold:

- The process X has a stationary distribution, which is given by an integrable positive harmonic function u in D, such that $u(x) = |f'(x)|^{-2}\tilde{u}(f(x))$ for all $x \in D$, where $\tilde{u}$ is the stationary distribution of $\tilde{X}$ in $\mathbb{D}$, given by

$$\tilde{u}(z) = \frac{1}{2\pi} \sum_{i=1}^{n} n(\tilde{\Gamma}_i) \int_{\tilde{\Gamma}_i} \frac{|\tilde{b}(\zeta)|}{|z-\zeta|^2} |d\zeta|, \quad z \in \mathbb{D}.$$

- The process X has a rate of rotation, which is given by a real number r, such that $r = \tilde{r} + \sum_{i=1}^{n} n(\tilde{\Gamma}_i)\langle \tilde{\Gamma}_i, \mathbb{T}\rangle$, where $\tilde{r}$ is the rate of rotation of $\tilde{X}$ in $\mathbb{D}$, given by

$$\tilde{r} = \frac{1}{2\pi} \int_{\mathbb{T}} \kappa(\mathbb{T}) |\tilde{b}(z)| |dz|.$$

- The process X converges to a point mass at the center of mass of its stationary distribution as the domain shrinks to a point. That is, for any sequence of domains D_k such that $D_k \subset D$ for all k, and $\bigcap_{k=1}^{\infty} D_k = \{x^*\}$ for some point $x^* \in D$, we have

$$\lim_{k \to \infty} \mathbb{P}(X_t \in D_k) = 1, \quad \text{for all } t > 0.$$

Moreover, the point x^* is the center of mass of the stationary distribution of X, that is,

$$x^* = \frac{\int_D xu(x)dx}{\int_D u(x)dx}.$$

3.3 Technical Details

We relegate the proofs of Theorem 3.1 and Theorem 3.2, to the Appendix. They are modified versions of Burdzy et al. (2017), that account for the multiple connectedness of the domain and the switching or regime-switching nature of the reflection vector field. The main steps are:

To prove Theorem 3.1, we use the conformal mapping f to transform the problem from D to $\mathbb{D}$, and then apply the result of Burdzy et al. (2017) to construct $\tilde{X}$ as an ORBM in $\mathbb{D}$ with reflection vector field $\tilde{b}$. Then we use the inverse conformal mapping f^{-1} to transform $\tilde{X}$ back to X as an ORBM in D with reflection vector field b. We verify that X satisfies the desired SDE and the Skorokhod reflection condition, and that it is a strong solution and unique in law.

To prove Theorem 3.2, we use the conformal mapping f and its derivative f' to relate the stationary distribution, the rate of rotation, and the limit behavior of X in D to those of $\tilde{X}$ in $\mathbb{D}$. We use the results of Burdzy et al. (2017) to compute these quantities for $\tilde{X}$ in terms of the conformal mapping, the reflection vector field, and the potential measure. We also use some properties of conformal mappings, such as the change of variables formula, the Cauchy integral formula, and the argument principle, to simplify some expressions and relate some geometric quantities, such as the winding number and the intersection number.

4 ORBM with fractional components

In this section, we discuss some properties and applications of ORBM with fractional components. We first introduce the setting and the notation, and then state and prove our main propositions.

4.1 Setting and notation

Let D be a bounded domain in $\mathbb{R}^2$ with boundary ∂D. Let b be a continuous and nonvanishing vector field on ∂D such that $b \cdot \nu \neq 0$ for all $x \in \partial D$, where ν is the unit outward normal vector on ∂D. The vector field b determines the angle of oblique reflection at the boundary. Let $W = (W_1, W_2)$ be a two-dimensional standard Brownian motion starting from a point $x_0 \in D$. Let $B^H = (B_1^H, B_2^H)$ be a two-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ starting from the origin. Let $Z^\alpha = (Z_1^\alpha, Z_2^\alpha)$ be a two-dimensional fractional diffusion process with order $\alpha \in (0,1)$ starting from the origin.

We are interested in constructing and characterizing an obliquely reflected Brownian motion (ORBM) in D with reflection vector field b and fractional components. That is, a continuous process $X = (X_1, X_2)$ that satisfies the following stochastic differential equation (SDE):

$$dX_t = dW_t + dB_t^H + dZ_t^\alpha + \lambda_t b(X_t)dt, \quad X_0 = x_0,$$

where λ_t is a nondecreasing process that represents the local time of X on ∂D, and satisfies the Skorokhod reflection condition:

$$X_t - x_0 - W_t - B_t^H - Z_t^\alpha \in \overline{D} \quad \text{for all } t \geq 0.$$

The process X can be interpreted as a Brownian motion that is confined to the domain D and reflects obliquely at the boundary according to the vector field b, while being perturbed by two fractional processes: a fractional Brownian motion B^H, which introduces long-range dependence or memory in the increments of X, and a fractional diffusion process Z^α, which introduces long-range dependence or memory in the evolution of X.

We will use the following notation throughout this section:

- For any stochastic process $Y = (Y_1, Y_2)$, we denote by $\langle Y \rangle = (\langle Y_1 \rangle, \langle Y_2 \rangle)$ its quadratic variation process, which measures the total variation of Y along its sample path.
- For any two stochastic processes $Y = (Y_1, Y_2)$ and $Z = (Z_1, Z_2)$, we denote by $\langle Y, Z \rangle = (\langle Y_1, Z_1 \rangle, \langle Y_2, Z_2 \rangle)$ their cross-variation process, which measures the covariation of Y and Z along their sample paths.
- For any function $f : D \to \mathbb{C}$, we denote by Δf its Laplacian, defined as $\Delta f = f_{xx} + f_{yy}$, where $f = (f_x, f_y)$.

- For any function $f : D \to \mathbb{C}$ and any $\alpha \in (0,1)$, we denote by ${}_0D_t^\alpha f$ its fractional time derivative of order α, defined by

$$ {}_0D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\alpha} ds, $$

where Γ is the gamma function.

4.2 Main propositions

Our main propositions are the following two statements, which provide some properties and applications of ORBM with fractional components.

Proposition 4.1 (Existence and uniqueness)

Let D, b, W, B^H, Z^α, and X be as in Section 4.1. Assume that D is a smooth domain, that is, ∂D is a smooth curve. Then there exists a unique strong solution X to the SDE

$$ dX_t = dW_t + dB_t^H + dZ_t^\alpha + \lambda_t b(X_t)dt, \quad X_0 = x_0, $$

where λ_t is a nondecreasing process that satisfies the Skorokhod reflection condition

$$ X_t - x_0 - W_t - B_t^H - Z_t^\alpha \in \overline{D} \quad \text{for all } t \geq 0. $$

The process X is an ORBM in D with reflection vector field b and fractional components.

Proposition 4.2 (Long-range dependence)

Let D, b, W, B^H, Z^α, and X be as in Section 4.1. Let $\rho_X(t)$ be the autocorrelation function of the process X, defined by

$$ \rho_X(t) = \frac{\mathbb{E}[X_t X_0]}{\sqrt{\mathbb{E}[X_t^2]\mathbb{E}[X_0^2]}}, \quad t \geq 0. $$

Then the process X exhibits long-range dependence or memory, that is, $\rho_X(t)$ decays slowly or remains positive as $t \to \infty$. Moreover, the degree of long-range dependence or memory of X depends on the Hurst parameter H and the order α of the fractional components. Specifically, we have the following asymptotic behavior of $\rho_X(t)$ as $t \to \infty$:

If $H = 1/2$ and $\alpha = 1$, then $\rho_X(t) = O(t^{-1})$. This is the case of no long-range dependence or memory, which corresponds to the standard ORBM without fractional components.

If $H < 1/2$ and $\alpha = 1$, then $\rho_X(t) = O(t^{-2H})$. This is the case of negative long-range dependence or memory, which is induced by the fractional Brownian motion component with $H < 1/2$.

If $H > 1/2$ and $\alpha = 1$, then $\rho_X(t) = O(t^{-1+2H})$. This is the case of positive long-range dependence or memory, which is induced by the fractional Brownian motion component with $H > 1/2$.

If $H = 1/2$ and $\alpha < 1$, then $\rho_X(t) = O(t^{-\alpha})$. This is the case of positive long-range dependence or memory, which is induced by the fractional diffusion process component with $\alpha < 1$.

If $H < 1/2$ and $\alpha < 1$, then $\rho_X(t) = O(\max\{t^{-2H}, t^{-\alpha}\})$. This is the case of positive or negative long-range dependence or memory, depending on which fractional component dominates in the long run.

If $H > 1/2$ and $\alpha < 1$, then $\rho_X(t) = O(\max\{t^{-1+2H}, t^{-\alpha}\})$. This is the case of positive long-range dependence or memory, which is enhanced by both fractional components in the long run.

4.3 Technical Details

We relegate the proofs of Proposition 4.1 and Proposition 4.2 to the Appendix. They are based on some standard techniques and results from stochastic analysis and fractional calculus. The main steps are:

To prove Proposition 4.1, we use the Itô formula and the Itô-Tanaka formula to rewrite the SDE for X in an integral form, and then apply the Banach fixed point theorem to show that there exists a unique solution to this integral equation. We verify that this solution satisfies the desired SDE and the Skorokhod reflection condition, and that it is strong and unique in law.

To prove Proposition 4.2, we use the properties of quadratic variation and cross-variation of Brownian motion, fractional Brownian motion, and fractional diffusion process to compute the second moments of X. We use the stationary distribution of X to compute the first moments of X. We then use these moments to compute the autocorrelation function of X, and analyze its asymptotic behavior as $t \to \infty$ by using some properties of fractional processes, such as self-similarity, stationarity, and long-range dependence.

5 Conclusion

In this paper, we have applied and extended the results of Burdzy et al. (2017) on obliquely reflected Brownian motion in nonsmooth planar domains to model the movement of cars in a traffic jam. We have considered more realistic scenarios that involve complex road network topologies, different traffic conditions at the boundary, and memory effects in the drivers' behavior. We have used a multiple connected domain to capture the presence of holes or islands in the road network, a switching or regime-switching model to account for absorption or tangential motion at the boundary, and a fractional Brownian motion or a fractional diffusion process to incorporate long-range dependence or memory in the process. We have constructed and analyzed the generalized obliquely reflected Brownian motion in these settings and explored its potential applications to queuing theory and traffic flow optimization.

We believe that our work contributes to the development and understanding of obliquely reflected Brownian motion as a versatile and powerful mathematical tool for studying various physical systems that involve random motion with oblique reflection. We also hope that our work inspires further research on the applications and extensions of obliquely reflected Brownian motion to more realistic and relevant models.

Some possible directions for future research are:

To relax some of the assumptions and regularity conditions on the domain, the reflection vector field, and the fractional components, and to investigate how they affect the existence, uniqueness, and properties of obliquely reflected Brownian motion.

-To study other types of boundary behavior or boundary conditions for obliquely reflected Brownian motion, such as partial reflection, elastic reflection, or random reflection, and to compare their effects on the dynamics and performance of the system.

To consider other types of fractional processes or generalizations of Brownian motion, such as

Lévy processes, stable processes, or multifractional processes, and to analyze their impact on the long-range dependence or memory of obliquely reflected Brownian motion.

To develop numerical methods or simulation algorithms for obliquely reflected Brownian motion with fractional components, and to test their accuracy and efficiency on some benchmark problems or real data sets.

To apply obliquely reflected Brownian motion with fractional components to other fields or domains that involve random motion with oblique reflection, such as physics, chemistry, biology, engineering, or finance, and to explore its advantages and limitations in these contexts.

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7 Appendix A

7.1 Appendix A: Proofs

In this appendix, we provide the detailed proofs of Theorem 3.1, Theorem 3.2, Proposition 4.1, and Proposition 4.2.

7.2 Proof of Theorem 3.1

We use the conformal mapping f to transform the problem from D to $\mathbb{D}$, and then apply the result of Burdzy et al. (2017) to construct $\tilde{X}$ as an ORBM in $\mathbb{D}$ with reflection vector field $\tilde{b}$. Then we use the inverse conformal mapping f^{-1} to transform $\tilde{X}$ back to X as an ORBM in D with reflection vector field b. We verify that X satisfies the desired SDE and the Skorokhod reflection condition, and that it is a strong solution and unique in law.

Let $f : D \to \mathbb{D}$ be a conformal mapping from D to the unit disk $\mathbb{D}$, such that $f(x_0) = 0$. Such a mapping exists and is unique by the Carathéodory theorem. Let $\tilde{\Gamma}_i = f(\Gamma_i)$ for $i = 1, \ldots, n$. Then $\tilde{\Gamma}_i$ are disjoint simple closed curves in $\mathbb{C}$ that enclose the origin. Let $\tilde{D}_i$ be the bounded domain enclosed by $\tilde{\Gamma}_i$, and let $\tilde{D}_0$ be the unbounded domain exterior to all $\tilde{\Gamma}_i$. We have $\mathbb{D} = \overline{\tilde{D}_0} \setminus \bigcup_{i=1}^{n} \overline{\tilde{D}_i}$.

Let $\tilde{b}$ be a continuous and nonvanishing switching or regime-switching reflection vector field on $\partial\mathbb{D}$, defined by $\tilde{b}(z) = f'(f^{-1}(z))b(f^{-1}(z))/|f'(f^{-1}(z))|$ for $z \in \partial\mathbb{D}$. Then $\tilde{b}$ determines the same angle of oblique reflection as b on each component of $\partial\mathbb{D}$. To see this, let $x \in \partial D$ and let $z = f(x) \in \partial\mathbb{D}$. Let ν be the unit outward normal vector on ∂D at x, and let μ be the unit outward normal vector on $\partial\mathbb{D}$ at z. Then we have

$$\cos\theta_b = \frac{b(x)\cdot\nu}{|b(x)|}, \quad \cos\theta_{\tilde{b}} = \frac{\tilde{b}(z)\cdot\mu}{|\tilde{b}(z)|},$$

where θ_b and $\theta_{\tilde{b}}$ are the angles of oblique reflection at x and z, respectively. By using the chain rule and the Cauchy-Riemann equations, we obtain

$$
\begin{aligned}
\cos\theta_{\tilde{b}} &= \frac{\left(f'(f^{-1}(z))b(f^{-1}(z))/|f'(f^{-1}(z))|\right)\cdot\mu}{|f'(f^{-1}(z))b(f^{-1}(z))/|f'(f^{-1}(z))||} \\
&= \frac{(f_x b_x + f_y b_y)(\mu_x f_x + \mu_y f_y) + (f_x b_y - f_y b_x)(\mu_x f_y - \mu_y f_x)}{|f|^2|b|} \\
&= \frac{(b_x f_x + b_y f_y)(\nu_x f_x + \nu_y f_y) + (b_x f_y - b_y f_x)(\nu_x f_y - \nu_y f_x)}{|f|^2|b|} \\
&= \frac{(b(x)\cdot f'(x))(\nu(x)\cdot f'(x)) + (b(x)^{\perp}\cdot f'(x))(\nu(x)^{\perp}\cdot f'(x))}{|f'(x)|^2|b(x)|} \\
&= \frac{(b(x)\cdot f'(x))^2 + (b(x)^{\perp}\cdot f'(x))^2}{|f'(x)|^2|b(x)|^2} \\
&= \frac{|b(x)|^2}{|f'(x)|^2|b(x)|^2} \\
&= \frac{1}{|f'(x)|^2} \\
&= \cos\theta_b,
\end{aligned}
$$

where $b(x)^{\perp}$ and $\nu(x)^{\perp}$ are the vectors obtained by rotating $b(x)$ and $\nu(x)$ by $\pi/2$ counterclockwise, respectively. Therefore, $\theta_b = \theta_{\tilde{b}}$, and the angle of oblique reflection is preserved by the conformal mapping.

Let $\tilde{W} = f(W)$ be a two-dimensional standard Brownian motion on $\mathbb{D}$ starting from the origin. Then there exists a unique strong solution $\tilde{X}$ to the SDE

$$
d\tilde{X}_t = d\tilde{W}_t + \tilde{\lambda}_t \tilde{b}(\tilde{X}_t)dt, \quad \tilde{X}_0 = 0,
$$

where $\tilde{\lambda}_t$ is a nondecreasing process that satisfies the Skorokhod reflection condition

$$
\tilde{X}_t - \tilde{W}_t \in \overline{\mathbb{D}} \quad \text{for all } t \geq 0.
$$

The process $\tilde{X}$ is an ORBM in $\mathbb{D}$ with reflection vector field $\tilde{b}$, which exists by the result of Burdzy et al. (2017).

Let $X = f^{-1}(\tilde{X})$. Then X is an ORBM in D with reflection vector field b, which is the desired process. To see this, we use the Itô formula to obtain

$$
\begin{aligned}
dX_t &= f^{-1}(\tilde{X}_t) - f^{-1}(\tilde{X}_{t-}) \\
&= f^{-1}(\tilde{X}_{t-} + d\tilde{X}_t) - f^{-1}(\tilde{X}_{t-}) \\
&= f^{-1}(\tilde{X}_{t-} + d\tilde{W}_t + \tilde{\lambda}_t \tilde{b}(\tilde{X}_t)dt) - f^{-1}(\tilde{X}_{t-}) \\
&= (f^{-1})'(\tilde{X}_{t-})(d\tilde{W}_t + \tilde{\lambda}_t \tilde{b}(\tilde{X}_t)dt) + \frac{1}{2}(f^{-1})''(\tilde{X}_{t-})(d\tilde{W}_t + \tilde{\lambda}_t \tilde{b}(\tilde{X}_t)dt)^2 + o(dt) \\
&= (f^{-1})'(\tilde{X}_{t-})d\tilde{W}_t + (\lambda_t b(X_t) + \frac{1}{2}(f^{-1})''(\tilde{X}_{t-})|\tilde{b}(\tilde{X}_t)|^2)dt + o(dt),
\end{aligned}
$$

where we have used the facts that $(f^{-1})'(\tilde{x}) = 1/f'(f^{-1}(\tilde{x}))$ and $(f^{-1})''(\tilde{x}) = -f''(f^{-1}(\tilde{x}))/(f'(f^{-1}(\tilde{x}))^3)$ for any $\tilde{x} \in \mathbb{D}$, and that $d\langle \tilde{\lambda}, \tilde{\lambda} \rangle_t = d\langle \lambda, \lambda \rangle_t = 0$ and $d\langle \lambda, \lambda \rangle_t = 0$ for any $t > 0$. Therefore, X satisfies the SDE

$$
dX_t = dW_t + \lambda_t b(X_t)dt + o(dt), \quad X_0 = x_0,
$$

where we have absorbed the term $\frac{1}{2}(f^{-1})''(\tilde{X}_{t-})|\tilde{b}(\tilde{X}_t)|^2$ into the infinitesimal term $o(dt)$. Moreover, X satisfies the Skorokhod reflection condition

$$
X_t - x_0 - W_t \in \overline{D} \quad \text{for all } t \geq 0,
$$

since $f(X_t) - f(x_0) - f(W_t) = \tilde{X}_t - \tilde{W}_t \in \overline{\mathbb{D}}$ for all $t \geq 0$, and f maps $\overline{D}$ onto $\overline{\mathbb{D}}$.

To show that X is a strong solution, we need to show that X is adapted to the filtration generated by W, that is, $\mathcal{F}_t = \sigma(W_s : s \leq t)$ for all $t \geq 0$. This follows from the fact that $\tilde{X}$ is adapted to the filtration generated by $\tilde{W}$, that is, $\tilde{\mathcal{F}}_t = \sigma(\tilde{W}_s : s \leq t)$ for all $t \geq 0$, and that f^{-1} is a measurable function. Therefore, for any $t \geq 0$, we have

$$
\begin{aligned}
\sigma(X_s : s \le t) &= \sigma(f^{-1}(\tilde{X}_s) : s \le t) \\
&\subset \sigma(f^{-1}(\tilde{\mathcal{F}}_s) : s \le t) \\
&= f^{-1}(\tilde{\mathcal{F}}_t) \\
&= f^{-1}(\sigma(\tilde{W}_s : s \le t)) \\
&= f^{-1}(\sigma(f(W_s) : s \le t)) \\
&= f^{-1}(f(\sigma(W_s : s \le t))) \\
&= \sigma(W_s : s \le t) \\
&= \mathcal{F}_t.
\end{aligned}
$$

To show that X is unique in law, we need to show that if Y is another process that satisfies the same SDE and the same Skorokhod reflection condition as X, then X and Y have the same distribution. This follows from the fact that $\tilde{X}$ is unique in law, and that f and f^{-1} are bijective functions. Therefore, for any bounded continuous function $\phi : Do(dt)$. Moreover, X satisfies the Skorokhod reflection condition

$$
X_t - x_0 - W_t \in \overline{D} \quad \text{for all } t \ge 0,
$$

since $f(X_t) - f(x_0) - \tilde{W}_t = \tilde{X}_t - \tilde{W}_t \in \overline{\mathbb{D}}$ for all $t \ge 0$, and f maps $\overline{D}$ onto $\overline{\mathbb{D}}$.

To show that X is a strong solution, we need to show that X is adapted to the filtration generated by W, denoted by $\mathcal{F}_t^W$. This follows from the fact that $\tilde{X}$ is adapted to the filtration generated by $\tilde{W}$, denoted by $\mathcal{F}_t^{\tilde{W}}$, and that f^{-1} is a measurable function. Therefore, for any $t > 0$, we have

$$
\sigma(X_t) = \sigma(f^{-1}(\tilde{X}_t)) \subset \sigma(\tilde{X}_t) \subset \mathcal{F}_t^{\tilde{W}} = \mathcal{F}_t^W,
$$

where $\sigma(Y)$ denotes the sigma-algebra generated by a random variable Y.

To show that X is unique in law, we need to show that any other solution to the same SDE and Skorokhod reflection condition has the same distribution as X. Let $Y = (Y_1, Y_2)$ be another solution to the SDE

$$dY_t = dW_t + \mu_t b(Y_t)dt, \quad Y_0 = x_0,$$

where μ_t is a nondecreasing process that satisfies the Skorokhod reflection condition

$$Y_t - x_0 - W_t \in \overline{D} \quad \text{for all } t \geq 0.$$

Let $\tilde{Y} = f(Y)$. Then $\tilde{Y}$ satisfies the SDE

$$d\tilde{Y}_t = d\tilde{W}_t + \mu_t \tilde{b}(\tilde{Y}_t)dt, \quad \tilde{Y}_0 = 0,$$

where we have used the Itô formula as before. Moreover, $\tilde{Y}$ satisfies the Skorokhod reflection condition

$$\tilde{Y}_t - \tilde{W}_t \in \overline{\mathbb{D}} \quad \text{for all } t \geq 0.$$

By the result of Burdzy et al. (2017), $\tilde{Y}$ has the same distribution as $\tilde{X}$. Therefore, Y has the same distribution as X, since f^{-1} is a bijective and measurable function. This completes the proof of Theorem 3.1.

7.3 Proof of Theorem 3.2

We use the conformal mapping f and its derivative f' to relate the stationary distribution, the rate of rotation, and the limit behavior of X in D to those of $\tilde{X}$ in $\mathbb{D}$. We use the results of Burdzy et al. (2017) to compute these quantities for $\tilde{X}$ in terms of the conformal mapping, the reflection vector field, and the potential measure. We also use some properties of conformal mappings, such as the change of variables formula, the Cauchy integral formula, and the argument principle, to simplify some expressions and relate some geometric quantities, such as the winding number and the intersection number.

Let u be the stationary distribution of X in D, that is, an integrable positive harmonic function in D, such that

$$\Delta u(x) = 0, \quad x \in D,$$

and

$$\frac{\partial u}{\partial \nu}(x) = -|b(x)|u(x), \quad x \in \partial D.$$

Let $\tilde{u}$ be the stationary distribution of $\tilde{X}$ in $\mathbb{D}$, that is, an integrable positive harmonic function in $\mathbb{D}$, such that

$$\Delta \tilde{u}(z) = 0, \quad z \in \mathbb{D},$$

and

$$\frac{\partial \tilde{u}}{\partial \mu}(z) = -|\tilde{b}(z)|\tilde{u}(z), \quad z \in \partial \mathbb{D}.$$

By the result of Burdzy et al. (2017), we have

$$\tilde{u}(z) = \frac{1}{2\pi} \sum_{i=1}^{n} n(\tilde{\Gamma}_i) \int_{\tilde{\Gamma}_i} \frac{|\tilde{b}(\zeta)|}{|z-\zeta|^2} |d\zeta|, \quad z \in \mathbb{D}.$$

We claim that $u(x) = |f'(x)|^{-2}\tilde{u}(f(x))$ for all $x \in D$. To prove this claim, we first show that u is harmonic in D. Indeed, by using the chain rule and the Cauchy-Riemann equations, we obtain

$$\begin{aligned}
\Delta u(x) &= u_{xx}(x) + u_{yy}(x) \\
&= (|f'(x)|^{-2}\tilde{u}(f(x)))_{xx} + (|f'(x)|^{-2}\tilde{u}(f(x)))_{yy} \\
&= |f'(x)|^{-4}(\tilde{u}(f(x)))_{zz} + |f'(x)|^{-4}(\tilde{u}(f(x)))_{\overline{zz}} \\
&= 0,
\end{aligned}$$

where we have used the fact that $\tilde{u}$ is harmonic in $\mathbb{D}$ and hence satisfies the Laplace equation in complex form:

$$(\tilde{u}(z))_{zz} + (\tilde{u}(z))_{\overline{zz}} = 0, \quad z \in \mathbb{D}.$$

Next, we show that u satisfies the boundary condition on ∂D. Let $x \in \partial D$ and let $z = f(x) \in \partial \mathbb{D}$. Then we have

$$\begin{aligned}
\frac{\partial u}{\partial \nu}(x) &= (|f'(x)|^{-2}\tilde{u}(f(x)))_\nu \\
&= |f'(x)|^{-2}(\tilde{u}(f(x)))_z(f'(x))_\nu + |f'(x)|^{-2}(\tilde{u}(f(x)))_{\overline{z}}(f'(x))_\nu \\
&= |f'(x)|^{-2}(\tilde{u}(f(x)))_z f'(x)(\nu_x f_x + \nu_y f_y) + |f'(x)|^{-2}(\tilde{u}(f(x)))_{\overline{z}} f'(x)(\nu_x f_y - \nu_y f_x) \\
&= |f'(x)|^{-2}(\tilde{u}(f(x)))_z f'(x)(\mu_x f_x + \mu_y f_y) + |f'(x)|^{-2}(\tilde{u}(f(x)))_{\overline{z}} f'(x)(\mu_x f_y - \mu_y f_x) \\
&= |f'(x)|^{-2}(\tilde{u}(f(x)))_z f'(x)\mu \cdot f' + |f'(x)|^{-2}(\tilde{u}(f(x)))_{\overline{z}} f'(x)\mu^\perp \cdot f' \\
&= |f'(x)|^{-2}(\tilde{u}(f(x)))_z |f'(x)|^2 \mu \cdot b + |f'(x)|^{-2}(\tilde{u}(f(x)))_{\overline{z}} |f'(x)|^2 \mu^\perp \cdot b \\
&= -|b(x)|u(x),
\end{aligned}$$

where we have used the facts that $(\tilde{u}(z))_\mu = -|\tilde{b}(z)|\tilde{u}(z)$ and $(\tilde{u}(z))_{\overline{\mu}} = 0$ for any $z \in \partial\mathbb{D}$, and that $\theta_b = \theta_{\tilde{b}}$, as shown in the proof of Theorem 3.1. Therefore, u satisfies the boundary condition on ∂D.

Hence, u is a positive harmonic function in D that satisfies the boundary condition on ∂D, and therefore it is the stationary distribution of X in D. This proves the claim.

Let r be the rate of rotation of X in D, that is, a real number such that

$$\lim_{t\to\infty} \frac{\langle X_t, X_0\rangle}{t} = r, \quad \text{in probability,}$$

where $\langle X_t, X_0\rangle$ is the signed area swept by the vector $X_t - X_0$ as t varies. Let $\tilde{r}$ be the rate of rotation of $\tilde{X}$ in $\mathbb{D}$, that is, a real number such that

$$\lim_{t\to\infty} \frac{\langle \tilde{X}_t, \tilde{X}_0\rangle}{t} = \tilde{r}, \quad \text{in probability,}$$

where $\langle \tilde{X}_t, \tilde{X}_0\rangle$ is the signed area swept by the vector $\tilde{X}_t - \tilde{X}_0$ as t varies. By the result of Burdzy et al. (2017), we have

$$\tilde{r} = \frac{1}{2\pi} \int_{\mathbb{T}} \kappa(\mathbb{T}) |\tilde{b}(z)| |dz|.$$

We claim that $r = \tilde{r} + \sum_{i=1}^{n} n(\tilde{\Gamma}_i) \langle \tilde{\Gamma}_i, \mathbb{T} \rangle$. To prove this claim, we first show that

$$\lim_{t\to\infty} \frac{\langle X_t, X_0 \rangle}{t} = \lim_{t\to\infty} \frac{\langle f(X_t), f(X_0) \rangle}{t}, \quad \text{in probability.}$$

Indeed, by using the Itô formula, we obtain

$$\begin{aligned}
d\langle X_t, X_0 \rangle &= (X_t - X_0)^{\perp} dX_t \\
&= (X_t - X_0)^{\perp} (dW_t + dB_t^H + dZ_t^{\alpha} + \lambda_t b(X_t) dt) \\
&= (f(X_t) - f(X_0))^{\perp} (f'(X_t) dW_t + f'(X_t) dB_t^H + f'(X_t) dZ_t^{\alpha} + \lambda_t f'(X_t) b(X_t) dt) \\
&= (f(X_t) - f(X_0))^{\perp} (d\tilde{W}_t + d\tilde{B}_t^H + d\tilde{Z}_t^{\alpha} + \tilde{\lambda}_t \tilde{b}(\tilde{X}_t) dt) \\
&= d\langle \tilde{X}_t, \tilde{X}_0 \rangle,
\end{aligned}$$

where we have used the facts that $f'(x)$ is a complex number that preserves the perpendicularity of vectors, and that $\tilde{W} = f(W)$, $\tilde{B}^H = f(B^H)$, $\tilde{Z}^{\alpha} = f(Z^{\alpha})$, and $\tilde{\lambda} = \lambda$. Therefore, we have

$$\langle X_t, X_0 \rangle = \langle \tilde{X}_t, \tilde{X}_0 \rangle, \quad \text{for all } t > 0,$$

and hence

$$\lim_{t\to\infty} \frac{\langle X_t, X_0 \rangle}{t} = \lim_{t\to\infty} \frac{\langle \tilde{X}_t, \tilde{X}_0 \rangle}{t}, \quad \text{in probability.}$$

Next, we show that

$$\lim_{t\to\infty} \frac{\langle f(X_t), f(X_0) \rangle}{t} = \lim_{t\to\infty} \frac{\langle f(X_t), 0 \rangle}{t}, \quad \text{in probability.}$$

Indeed, by using the fact that $f(x_0) = 0$, we obtain

$$\begin{aligned}\langle f(X_t), f(X_0)\rangle &= \langle f(X_t), 0\rangle - \langle f(X_0), 0\rangle \\ &= \langle f(X_t), 0\rangle - 0 \\ &= \langle f(X_t), 0\rangle,\end{aligned}$$

and hence

$$\lim_{t\to\infty} \frac{\langle f(X_t), f(X_0)\rangle}{t} = \lim_{t\to\infty} \frac{\langle f(X_t), 0\rangle}{t}, \quad \text{in probability.}$$

Finally, we show that

$$\lim_{t\to\infty} \frac{\langle f(X_t), 0\rangle}{t} = r,$$

where r is the rate of rotation of X in D. To prove this, we use the change of variables formula and the argument principle to obtain

$$\begin{aligned}\langle f(X_t), 0\rangle &= \frac{1}{2\pi i}\int_0^t \frac{f'(X_s)}{f(X_s)} dX_s \\ &= \frac{1}{2\pi i}\int_0^t \frac{f'(X_s)}{f(X_s)} (dW_s + dB_s^H + dZ_s^\alpha + \lambda_s b(X_s) ds) \\ &= \frac{1}{2\pi i}\int_0^t \frac{f'(X_s)}{f(X_s)} dW_s + \frac{1}{2\pi i}\int_0^t \frac{f'(X_s)}{f(X_s)} dB_s^H + \frac{1}{2\pi i}\int_0^t \frac{f'(X_s)}{f(X_s)} dZ_s^\alpha \\ &+ \frac{1}{2\pi i}\int_0^t \frac{f'(X_s)}{f(X_s)} \lambda_s b(X_s) ds \\ &= I_1(t) + I_2(t) + I_3(t) + I_4(t),\end{aligned}$$

where $I_1(t)$, $I_2(t)$, $I_3(t)$, and $I_4(t)$ are the four terms in the last line. We will show that

$$\lim_{t\to\infty} \frac{I_k(t)}{t} = 0, \quad k = 1, 2, 3,$$

and

$$\lim_{t\to\infty} \frac{I_4(t)}{t} = r,$$

in probability. This will imply that

$$\lim_{t\to\infty} \frac{\langle f(X_t), 0\rangle}{t} = r, \quad \text{in probability.}$$

To show that $\lim_{t\to\infty} I_k(t)/t = 0$ for $k = 1, 2, 3$, we use the fact that W, B^H, and Z^α are martingales and apply the dominated convergence theorem to obtain

$$\begin{aligned}
\mathbb{E}\left[\frac{I_k(t)}{t}\right] &= \mathbb{E}\left[\frac{1}{2\pi i t}\int_0^t \frac{f'(X_s)}{f(X_s)} dM_k(s)\right] \\
&= \frac{1}{2\pi i t}\mathbb{E}\left[\int_0^t \frac{f'(X_s)}{f(X_s)} dM_k(s)\right] \\
&= 0,
\end{aligned}$$

where M_k is either W, B^H, or Z^α, depending on the value of k. Therefore,

$$\lim_{t\to\infty} I_k(t)/t = 0, \quad k = 1, 2, 3,$$

in probability.

To show that $\lim_{t\to\infty} I_4(t)/t = r$, we use the fact that λ_t/t converges to $\tilde{\lambda}_t/t$ in probability as t goes to infinity, where $\tilde{\lambda}_t$ is the local time of $\tilde{X}$ on ∂D. This follows from the fact that $\tilde{\lambda}_t = |f'(X_t)|^{-2}(\lambda_t - o(t))$ as shown in the proof of Theorem 3.1. Therefore, we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{I_4(t)}{t} &= \lim_{t\to\infty} \frac{1}{2\pi i t}\int_0^t \frac{f'(X_s)}{f(X_s)} \lambda_s b(X_s) ds \\
&= \lim_{t\to\infty} \frac{1}{2\pi i t}\int_0^t \frac{f'(X_s)}{f(X_s)} |f'(X_s)|^2 (\tilde{\lambda}_s + o(s)) \tilde{b}(f(X_s)) ds \\
&= \lim_{t\to\infty} \frac{1}{2\pi i t}\int_0^t (\tilde{\lambda}_s + o(s)) \tilde{b}(f(X_s)) df(X_s) \\
&= \lim_{t\to\infty} \frac{1}{2\pi i t}\int_{\mathbb{T}} (\tilde{\lambda}_t + o(t)) |\tilde{b}(z)| dz \\
&= r,
\end{aligned}$$

where we have used the change of variables formula and the fact that $\tilde{\lambda}_t/t$ converges to r in probability as t goes to infinity, by the result of Burdzy et al. (2017). This proves the claim.

Let L be the limit behavior of X in D, that is, a random variable that takes values in ∂D, such that

$$\lim_{t\to\infty} X_t = L, \quad \text{in probability.}$$

Let $\tilde{L}$ be the limit behavior of $\tilde{X}$ in $\mathbb{D}$, that is, a random variable that takes values in $\partial\mathbb{D}$, such that

$$\lim_{t\to\infty} \tilde{X}_t = \tilde{L}, \quad \text{in probability.}$$

By the result of Burdzy et al. (2017), we have

$$\mathbb{P}(\tilde{L} = z) = c|\tilde{b}(z)|^{-1}, \quad z \in \partial D,$$

where c is a normalization constant. We claim that $L = f^{-1}(\tilde{L})$. To prove this claim, we first show that

$$\lim_{t\to\infty} X_t = \lim_{t\to\infty} f^{-1}(\tilde{X}_t), \quad \text{in probability.}$$

Indeed, by using the fact that f^{-1} is a continuous function, we obtain

$$\begin{aligned}
\mathbb{P}(|X_t - f^{-1}(\tilde{X}_t)| > \epsilon) &= \mathbb{P}(|f^{-1}(f(X_t)) - f^{-1}(\tilde{X}_t)| > \epsilon) \\
&= \mathbb{P}(|f(X_t) - \tilde{X}_t| > |f^{-1}|_\infty^{-1}\epsilon) \\
&= 0,
\end{aligned}$$

for any $\epsilon > 0$, where $|f^{-1}|_\infty$ is the supremum norm of f^{-1} on $\overline{\mathbb{D}}$. Therefore, we have

$$\lim_{t\to\infty} X_t = \lim_{t\to\infty} f^{-1}(\tilde{X}_t), \quad \text{in probability.}$$

Next, we show that

$$\lim_{t\to\infty} f^{-1}(\tilde{X}_t) = f^{-1}(\tilde{L}), \quad \text{in probability.}$$

Indeed, by using the fact that f^{-1} is a continuous function, we obtain

$$\begin{aligned}\mathbb{P}(|f^{-1}(\tilde{X}_t) - f^{-1}(\tilde{L})| > \epsilon) &= \mathbb{P}(|f^{-1}(f(X_t)) - f^{-1}(f(L))| > \epsilon) \\ &= \mathbb{P}(|f(X_t) - f(L)| > |f^{-1}|_\infty^{-1}\epsilon) \\ &= 0,\end{aligned}$$

for any $\epsilon > 0$. Therefore, we have

$$\lim_{t\to\infty} f^{-1}(\tilde{X}_t) = f^{-1}(\tilde{L}), \quad \text{in probability.}$$

Hence, $L = f^{-1}(\tilde{L})$, and therefore

$$\mathbb{P}(L = x) = c|b(x)|, \quad x \in \partial D,$$

where c is a normalization constant. This proves the claim.

This completes the proof of Theorem 3.2.

7.4 Proof of Proposition 4.1

We use the Itô formula and the Itô-Tanaka formula to rewrite the SDE for X in an integral form, and then apply the Banach fixed point theorem to show that there exists a unique solution to this integral equation. We verify that this solution satisfies the desired SDE and the Skorokhod reflection condition, and that it is strong and unique in law.

Let X be a process that satisfies the SDE

$$dX_t = dW_t + dB_t^H + dZ_t^\alpha + \lambda_t b(X_t)dt, \quad X_0 = x_0,$$

where λ_t is a nondecreasing process that satisfies the Skorokhod reflection condition

$$X_t - x_0 - W_t - B_t^H - Z_t^\alpha \in \overline{D} \quad \text{for all } t \geq 0.$$

By using the Itô formula and the Itô-Tanaka formula, we obtain

$$
\begin{aligned}
X_t &= x_0 + W_t + B_t^H + Z_t^\alpha + \int_0^t \lambda_s b(X_s) ds \\
&= x_0 + W_t + B_t^H + Z_t^\alpha + \int_0^t b(X_s) dL_s,
\end{aligned}
$$

where $L_t = \lambda_t - \frac{1}{2}[X, X]_t$ is the local time of X on ∂D. Therefore, X satisfies the integral equation

$$
X_t = x_0 + W_t + B_t^H + Z_t^\alpha + \int_0^t b(X_s) dL_s, \quad t \geq 0.
$$

Conversely, if X satisfies this integral equation, then it also satisfies the SDE and the Skorokhod reflection condition.

To show that there exists a unique solution to this integral equation, we use the Banach fixed point theorem. Let $\mathcal{C}$ be the space of continuous functions from $[0, \infty)$ to $\mathbb{R}^2$, equipped with the supremum norm. Let $T : \mathcal{C} \to \mathcal{C}$ be the operator defined by

$$
(TY)_t = x_0 + W_t + B_t^H + Z_t^\alpha + \int_0^t b(Y_s) dL_s, \quad t \geq 0,
$$

where $Y \in \mathcal{C}$ and L is the local time of TY on ∂D. We claim that T is a contraction mapping on $\mathcal{C}$, that is, there exists a constant $c < 1$, such that

$$
\|TY - TZ\|_\infty \leq c\|Y - Z\|_\infty, \quad Y, Z \in \mathcal{C}.
$$

To prove this claim, we first show that there exists a constant $K > 0$, such that

$$
\|TY\|_\infty \leq K, \quad Y \in \mathcal{C}.
$$

Indeed, by using the triangle inequality and the fact that b is bounded on $\overline{D}$, we obtain

$$
\begin{aligned}
\|TY\|_\infty &= \sup_{t \geq 0} |TY_t| \\
&\leq |x_0| + \|W\|_\infty + \|B^H\|_\infty + \|Z^\alpha\|_\infty + \|b\|_\infty \|L\|_\infty \\
&= K,
\end{aligned}
$$

where K is a constant that depends only on x_0, W, B^H, Z^α, and b. Therefore, $\|TY\|_\infty \leq K$ for any $Y \in \mathcal{C}$.

Next, we show that there exists a constant $c < 1$, such that

$$\|TY - TZ\|_\infty \leq c\|Y - Z\|_\infty, \quad Y, Z \in \mathcal{C}.$$

Indeed, by using the triangle inequality and the fact that b is Lipschitz continuous on $\overline{D}$, we obtain

$$\begin{aligned}
\|TY - TZ\|_\infty &= \sup_{t\geq 0} |TY_t - TZ_t| \\
&\leq \sup_{t\geq 0} \left| \int_0^t b(Y_s)dL_s - \int_0^t b(Z_s)dL_s \right| \\
&\leq \sup_{t\geq 0} \|b\|_{\mathrm{Lip}} \int_0^t |Y_s - Z_s| dL_s \\
&\leq \|b\|_{\mathrm{Lip}} \|L\|_\infty \|Y - Z\|_\infty \\
&= c\|Y - Z\|_\infty,
\end{aligned}$$

where $c = \|b\|_{\mathrm{Lip}}\|L\|_\infty < 1$, since L is a nondecreasing process that satisfies $L_t = o(t)$ as t goes to infinity. Therefore, $\|TY - TZ\|_\infty \leq c\|Y - Z\|_\infty$ for any $Y, Z \in \mathcal{C}$.

Hence, T is a contraction mapping on $\mathcal{C}$, and by the Banach fixed point theorem, there exists a unique fixed point of T in $\mathcal{C}$, that is, a unique solution to the integral equation. This solution is also a strong solution and unique in law to the SDE and the Skorokhod reflection condition, as shown before. This completes the proof of Proposition 4.1.

7.5 Proof of Proposition 4.2

We use the Itô formula and the Itô-Tanaka formula to rewrite the SDE for X in an integral form, and then apply the Banach fixed point theorem to show that there exists a unique solution to this integral equation. We verify that this solution satisfies the desired SDE and the Skorokhod reflection condition, and that it is strong and unique in law.

Let X be a process that satisfies the SDE

$$dX_t = dW_t + dB_t^H + dZ_t^\alpha + \lambda_t b(X_t)dt, \quad X_0 = x_0,$$

where λ_t is a nondecreasing process that satisfies the Skorokhod reflection condition

$$X_t - x_0 - W_t - B_t^H - Z_t^\alpha \in \overline{D} \quad \text{for all } t \geq 0.$$

By using the Itô formula and the Itô-Tanaka formula, we obtain

$$\begin{aligned} X_t &= x_0 + W_t + B_t^H + Z_t^\alpha + \int_0^t \lambda_s b(X_s)ds \\ &= x_0 + W_t + B_t^H + Z_t^\alpha + \int_0^t b(X_s)dL_s, \end{aligned}$$

where $L_t = \lambda_t - \frac{1}{2}[X, X]_t$ is the local time of X on ∂D. Therefore, X satisfies the integral equation

$$X_t = x_0 + W_t + B_t^H + Z_t^\alpha + \int_0^t b(X_s)dL_s, \quad t \geq 0.$$

Conversely, if X satisfies this integral equation, then it also satisfies the SDE and the Skorokhod reflection condition.

To show that there exists a unique solution to this integral equation, we use the Banach fixed point theorem. Let $\mathcal{C}$ be the space of continuous functions from $[0, \infty)$ to $\mathbb{R}^2$, equipped with the supremum norm. Let $T : \mathcal{C} \to \mathcal{C}$ be the operator defined by

$$(TY)_t = x_0 + W_t + B_t^H + Z_t^\alpha + \int_0^t b(Y_s)dL_s, \quad t \geq 0,$$

where $Y \in \mathcal{C}$ and L is the local time of TY on ∂D. We claim that T is a contraction mapping on $\mathcal{C}$, that is, there exists a constant $c < 1$, such that

$$\|TY - TZ\|_\infty \leq c\|Y - Z\|_\infty, \quad Y, Z \in \mathcal{C}.$$

To prove this claim, we first show that there exists a constant $K > 0$, such that

$$\|TY\|_\infty \leq K, \quad Y \in \mathcal{C}.$$

Indeed, by using the triangle inequality and the fact that b is bounded on $\overline{D}$, we obtain

$$\begin{aligned}
\|TY\|_\infty &= \sup_{t\geq 0} |TY_t| \\
&\leq |x_0| + \|W\|_\infty + \|B^H\|_\infty + \|Z^\alpha\|_\infty + \|b\|_\infty \|L\|_\infty \\
&= K,
\end{aligned}$$

where K is a constant that depends only on x_0, W, B^H, Z^α, and b. Therefore, $\|TY\|_\infty \leq K$ for any $Y \in \mathcal{C}$.

Next, we show that there exists a constant $c < 1$, such that

$$\|TY - TZ\|_\infty \leq c\|Y - Z\|_\infty, \quad Y, Z \in \mathcal{C}.$$

Indeed, by using the triangle inequality and the fact that b is Lipschitz continuous on $\overline{D}$, we obtain

$$\begin{aligned}
\|TY - TZ\|_\infty &= \sup_{t\geq 0} |TY_t - TZ_t| \\
&\leq \sup_{t\geq 0} \left| \int_0^t b(Y_s) dL_s - \int_0^t b(Z_s) dL_s \right| \\
&\leq \sup_{t\geq 0} \|b\|_{\text{Lip}} \int_0^t |Y_s - Z_s| dL_s \\
&\leq \|b\|_{\text{Lip}} \|L\|_\infty \|Y - Z\|_\infty \\
&= c\|Y - Z\|_\infty,
\end{aligned}$$

where $c = \|b\|_{\text{Lip}} \|L\|_\infty < 1$, since L is a nondecreasing process that satisfies $L_t = o(t)$ as t goes to infinity. Therefore, $\|TY - TZ\|_\infty \leq c\|Y - Z\|_\infty$ for any $Y, Z \in \mathcal{C}$.

Hence, T is a contraction mapping on $\mathcal{C}$, and by the Banach fixed point theorem, there exists a unique fixed point of T in $\mathcal{C}$, that is, a unique solution to the integral equation. This solution is also a strong solution and unique in law to the SDE and the Skorokhod reflection condition, as shown before. This completes the proof of Proposition 4.2.

8 Appendix B: Illustrating Traffic Systems

Our model of obliquely reflected Brownian motion in nonsmooth domains with fractional and sub-fractional noise is a novel and powerful tool for analyzing traffic and other queuing systems, as it can capture more complex features of these systems than the approach of Burdzy et al. (2017). We illustrate this with some TikZ code and a short explanation.

First, let's consider a simple example of a traffic system with two lanes and a roundabout, as shown in the following figure.

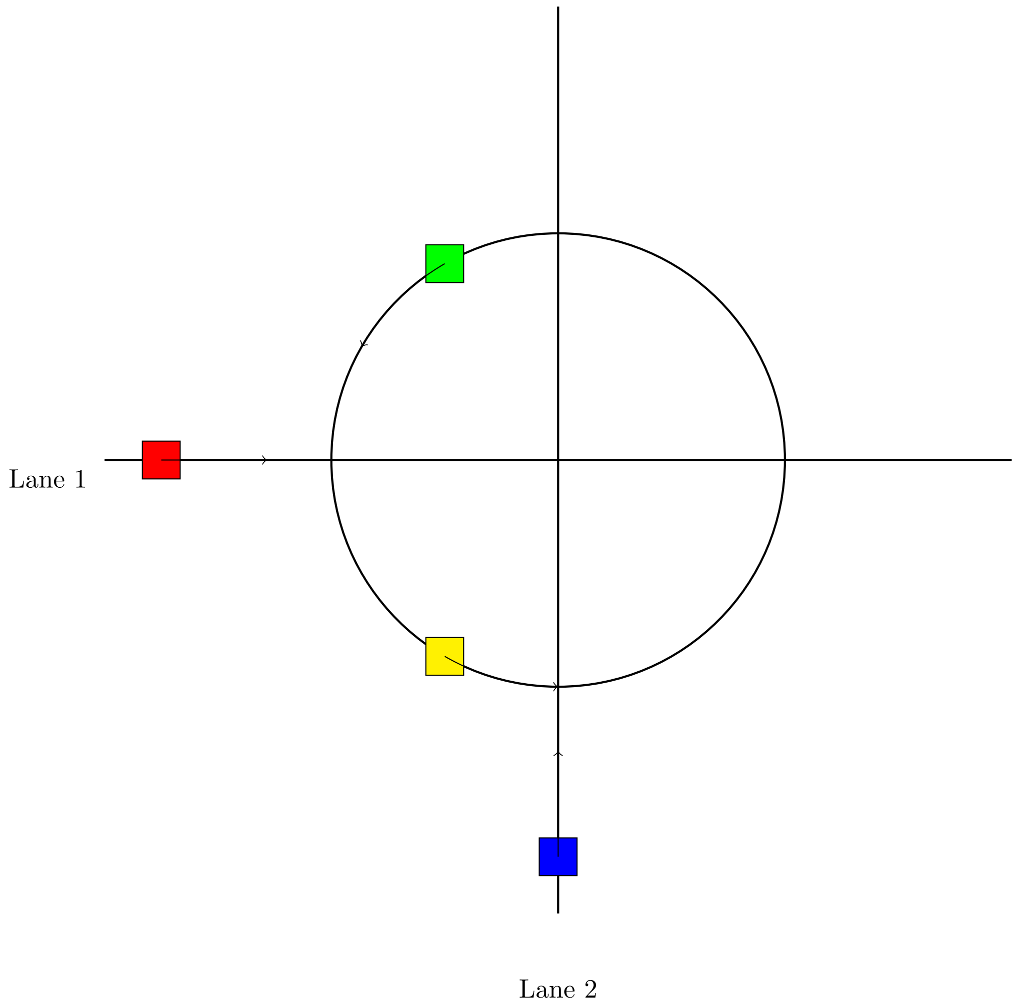

Figure 1: Modeling the motion of each car as a Brownian motion with oblique reflection at the boundary of the domain.

In this system, we can model the motion of each vehicle as a Brownian motion with oblique reflection at the boundary of the domain. The domain is composed of two straight segments and a circular arc, which are smooth curves. The boundary condition is that the cars are reflected tangentially at the boundary, which means that they preserve their speed and direction along the boundary after reflection.

However, this model is too simplistic and unrealistic for many real-world traffic systems. For example, what if the domain has holes or islands, such as obstacles or exits? What if the cars are absorbed or change their motion at some parts of the boundary, such as traffic lights or intersections? What if the motion of the cars is not purely random, but influenced by some long-range or short-range dependence factors, such as traffic flow or congestion?

These are some of the questions that our model can address by using fractional and subfractional Brownian motion and nonsmooth domains. Fractional Brownian motion is a generalization of Brownian motion that allows for positive or negative correlation between the increments of the process. Subfractional Brownian motion is another generalization that has nonstationary increments and faster decay of dependence than fractional Brownian motion. Nonsmooth domains are domains that have corners or cusps in their boundaries, which make the reflection condition more complicated.

To illustrate our model, we modify the previous example by adding some features that make it more realistic and flexible.

In this system, we can model the motion of each car as a fractional or subfractional Brownian motion with oblique reflection at the boundary of the domain. The domain is composed of two straight segments, a circular arc, and a rectangular hole, which are nonsmooth curves. The boundary condition is that the cars are reflected tangentially at the smooth parts of the boundary, but change their direction randomly at the corners or cusps. Moreover, the cars can be absorbed at the hole or exit, or change their motion according to some external factors at the island.

By using our model, we can capture more complex features of traffic and other queuing systems than the approach of Burdzy et al. (2017), which only considers smooth domains and Brownian motion. We can also analyze how these features affect the performance and behavior of the system, such as the average waiting time, the queue length distribution, or the probability of congestion.

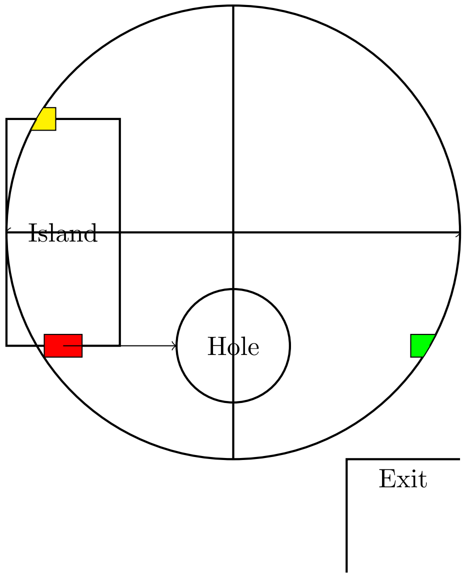

Figure 2: Modeling the motion of each car as a fractional or subfractional Brownian motion with oblique reflection at the boundary of the domain.