Prices and the Generalized Production Possibilities Frontier

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Abstract

In this paper, we propose a generalization of the Production Possibilities Frontier (PPF) that incorporates prices. Price is the difference in marginal utility or cost between two agents, which drives the exchange of goods or services. The slope of the PPF could be seen as the price ratio between two goods or services, which determines how much of one good or service must be given up to produce more of another good or service. Prices are taken as given and provided by an institution. We show that the standard PPF, derived from Pareto Efficiency, has some limitations in capturing complex scenarios of interest to both microeconomists and macroeconomists. We introduce a more general PPF, that can accommodate more shapes and cases and unpack scenarios ranging from production functions, utility functions, social welfare functions, budget constraints, and technological change. We also prove that the standard PPF is a special case of the general PPF. The key insight is a novel relationship between three variables: prices, or the difference in value between two goods or services; trade, or the rate of exchange; and the amount of utility that is lost as opportunity cost per unit of good or service that passes through a point in a market.

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Contents

1 Introduction

The Production Possibilities Frontier (PPF) is a fundamental concept in economics that illustrates the trade-offs and opportunity costs of producing different combinations of goods or services with given resources. The PPF is usually derived from the assumption of Pareto Efficiency, which states that an allocation is efficient if there is no other allocation that makes at least one agent better off without making any other agent worse off. The shape of the PPF depends on the production technology and the preferences of the agents involved. The slope of the PPF represents the marginal rate of transformation (MRT), which is the amount of one good that must be given up to produce one more unit of another good.

The PPF has been widely used in both microeconomics and macroeconomics to analyze various issues such as comparative advantage, trade, growth, income distribution, and energy economics[1]. However, the standard PPF has some limitations in capturing complex scenarios that involve multiple agents, multiple goods, multiple technologies, and multiple constraints. For instance, the standard PPF cannot account for externalities, market failures, public goods, increasing returns to scale, indivisibilities, or non-convexities.

In this paper, we propose a generalization of the Production Possibilities Frontier (PPF) that incorporates prices. Price is the difference in marginal utility or cost between two agents, which drives the exchange of goods or services. The slope of the PPF could be seen as the price ratio between two goods or services, which determines how much of one good or service must be given up to produce more of another good or service. Prices are taken as given and provided by an institution.

Consider a standard PPF where that the economy produces two goods, X and Y, with the same production function. The PPF is given by

$$P(Y) = \{(x, y) \in \mathbb{R}^2 : x = AK^{\alpha}L^{1-\alpha} - y\}$$

To represent the generalized PPF model that allows for different production functions and other

[1]The sheer volume of studies that inherently rely on a PPF are far too many to list and the concept is part of the canon.

novel features, we propose to integrate pricing into the PPF. The generalized PPF is now given by

$$P'(Y) = \{(x, y) \in \mathbb{R}^2 : x = f_1(\bar{K}_1, \bar{L}_1), y = f_2(\bar{K}_2, \bar{L}_2), \eta(x, y) = \max_{(x',y') \in \mathbb{R}^2} \eta(x', y')\}$$

. By integrating the PPF with prices we open the floodgates to integrate the entire toolkit into the PPF in one fell swoop.

Our proposed generalization of the PPF based on the analogy between voltage in engineering and price in economics. The analogy is simple. Voltage is the difference in electric potential energy between two points, which drives the electric current. Price is the difference in marginal utility or cost between two agents, which drives the exchange of goods or services. We exploit the fact that both voltage and price reflect the opportunity cost of using a resource in one way versus another. The slope of the PPF is thus seen as the voltage or price ratio between two goods or services, which determines how much of one good or service must be given up to produce more of another good or service. The key insight is an economic version of Ohm's Law: a novel relationship between three variables: (1) prices or the difference in value between two goods or services; (2) trade, or the rate of exchange; and (3) the amount of utility that is lost as opportunity cost per unit of good or service that passes through a point in a market. Ohm's Law is a foundational result in computer engineering and other fields. This approach is distinct from recent computing contributions to economics, which emphasize the sub-field software (such as machine learning) and not hardware[2].

We show that by using the familiar concept of pricing, we can derive a more general PPF, denoted by $P'(Y)$, that can accommodate more shapes and cases than the standard PPF $P(Y)$. We also prove that the standard PPF is a special case of the general PPF. We illustrate the applications and implications of our general PPF for various topics in economics such as production functions, utility functions, social welfare functions, budget constraints, and technological change.

Our general PPF model is a subset of the standard PPF model in the sense that any point on our general PPF is also on the standard PPF, but not vice versa. However, our general PPF model can accommodate more shapes and cases than the standard PPF model in the sense that our general PPF can have different slopes and curvatures depending on what we shall call the resistance

[2] A few examples of the thriving machine learning research in economics include Chalfin et al (2016), Mullainathan, and Obermeyer (2017), Athey and Imbens (2019), Mullainathan, and Obermeyer (2022), and Aiken et al (2022)

function, while the standard PPF can only have a constant negative slope and a constant concavity. Therefore, our general PPF model is more flexible and realistic than the standard PPF model.

The rest of the paper is organized as follows. Section 2 reviews the literature on the PPF and its extensions. Section 3 presents our general PPF model and its properties. Section 4 discusses some examples and applications of our general PPF. Section 5 concludes with some directions for future research. The Appendix presents additional details.

2 Literature Review

The PPF was introduced by Robbins (1932) as a graphical representation of the production possibilities of a society given its resources and technology. It was further developed by Samuelson (1947) and Debreu (1951) as a tool for analyzing general equilibrium and welfare economics, and extended by Solow (1956) and Swan (1956) to incorporate economic growth and technological change. Dasgupta and Heal (1974) and Hartwick (1977) to included natural resources studies into the framework.

Most recently, the PPF was generalized by Arrow et al. (2012) to allow for multiple agents, multiple goods, multiple technologies, and multiple constraints. However, none of the existing models of the PPF can account for the role of price in determining the shape and slope of the frontier. Our paper fills this gap by proposing a new model of the PPF based on the analogy between voltage and price. We show that our model can capture more complex scenarios than the standard PPF and has important implications for economic analysis.

3 General PPF Model

We present our general PPF model and its properties now. The model proceeds as follows. We consider a system with function $f : \mathbb{R}^n \to \mathbb{R}^m$, where X is a compact space of feasible decisions (including allocations of time and endowment goods) in the metric space $\mathbb{R}^n$, and Y is the feasible set of criterion vectors (say, final goods and services) in $\mathbb{R}^m$, such that $Y = \{y \in \mathbb{R}^m : y = f(x), x \in X\}$.

We assume that the preferred directions of criteria values are known so that more of any good in Y is better. A point $y'' \in \mathbb{R}^m$ strictly dominates another point $y' \in \mathbb{R}^m$, written as $y'' > y'$, means that for each element index i $y''_i \geq y'_i$ and there is at least one element j such that $y''_j > y'_j$.

The Pareto frontier is thus written as: $P(Y) = \{y' \in Y : \{y'' \in Y : y'' > y', y'' \neq y'\} = \emptyset\}$.

We define voltage as the difference in electric potential energy between two points, which drives the electric current. We assume that there is a voltage source that supplies a constant voltage V to the system. The voltage source can be interpreted as an external agent or institution that sets the price or incentive for producing or consuming the goods or services in Y.

We define resistance as the opposition to the flow of electric current, which depends on the characteristics of the system. We assume that there is a resistance function $R : Y \to \mathbb{R}_+$ that maps each point in Y to a positive real number. The resistance function can be interpreted as the cost or difficulty of producing or consuming the goods or services in Y.

We define current as the rate of flow of electric charge, which depends on the voltage and the resistance. We assume that there is a current function $I : Y \to \mathbb{R}_+$ that maps each point in Y to a positive real number. The current function can be interpreted as the quantity or quality of the goods or services in Y. We also assume that the current function satisfies Ohm's law, which states that $I(y) = \frac{V}{R(y)}$ for any $y \in Y$.

We define power as the rate of doing work, which depends on the voltage and the current. We assume that there is a power function $P : Y \to \mathbb{R}_+$ that maps each point in Y to a positive real number. The power function can be interpreted as the utility or benefit of producing or consuming the goods or services in Y. We also assume that the power function satisfies Joule's law, which states that $P(y) = VI(y)$ for any $y \in Y$.

We define efficiency as the ratio of output power to input power, which measures how well the system converts electric energy into useful work. We assume that there is an efficiency function $\eta : Y \to [0, 1]$ that maps each point in Y to a number between 0 and 1. The efficiency function can be interpreted as the degree of Pareto optimality or social welfare of producing or consuming the goods or services in Y. We also assume that the efficiency function satisfies $\eta(y) = \frac{P(y)}{V^2/R(y)}$ for any $y \in Y$.

We define our general PPF as follows:

$$P'(Y) = \{y' \in Y : \eta(y') = \max_{y \in Y} \eta(y)\}$$

3.1 Properties of the General PPF

We show that our general PPF has the following properties:

1. It is a subset of the standard PPF, i.e., $P'(Y) \subseteq P(Y)$.
2. It is convex if and only if the resistance function is convex, i.e., $\forall y_1, y_2 \in Y, \forall t \in [0,1], R(ty_1 + (1-t)y_2) \leq tR(y_1) + (1-t)R(y_2)$.
3. It is linear if and only if the resistance function is linear, i.e., $\forall y_1, y_2 \in Y, \forall t \in [0,1], R(ty_1 + (1-t)y_2) = tR(y_1) + (1-t)R(y_2)$.
4. It is concave if and only if the resistance function is concave, i.e., $\forall y_1, y_2 \in Y, \forall t \in [0,1], R(ty_1 + (1-t)y_2) \geq tR(y_1) + (1-t)R(y_2)$.
5. It is non-convex if and only if the resistance function is non-convex, i.e., $\exists y_1, y_2 \in Y, \exists t \in [0,1], R(ty_1 + (1-t)y_2) > tR(y_1) + (1-t)R(y_2)$.
6. It has a negative slope if and only if the resistance function is increasing, i.e., $\forall y_1, y_2 \in Y, y_1 > y_2 \implies R(y_1) > R(y_2)$.
7. It has a zero slope if and only if the resistance function is constant, i.e., $\forall y \in Y, R(y) = c$ for some $c \in \mathbb{R}_+$.
8. It has a positive slope if and only if the resistance function is decreasing, i.e., $\forall y_1, y_2 \in Y, y_1 > y_2 \implies R(y_1) < R(y_2)$.

Proofs are in the Appendix.

3.2 Illustration of the General PPF

The diagram shows the trade-off and efficiency level between output and input in an economy that uses a production function and a resistance function. The output is measured on the vertical axis and the input is measured on the horizontal axis. The production function determines the maximum output that can be produced for a given input, and the resistance function determines the opposition to the flow of current in the circuit that represents the economy.

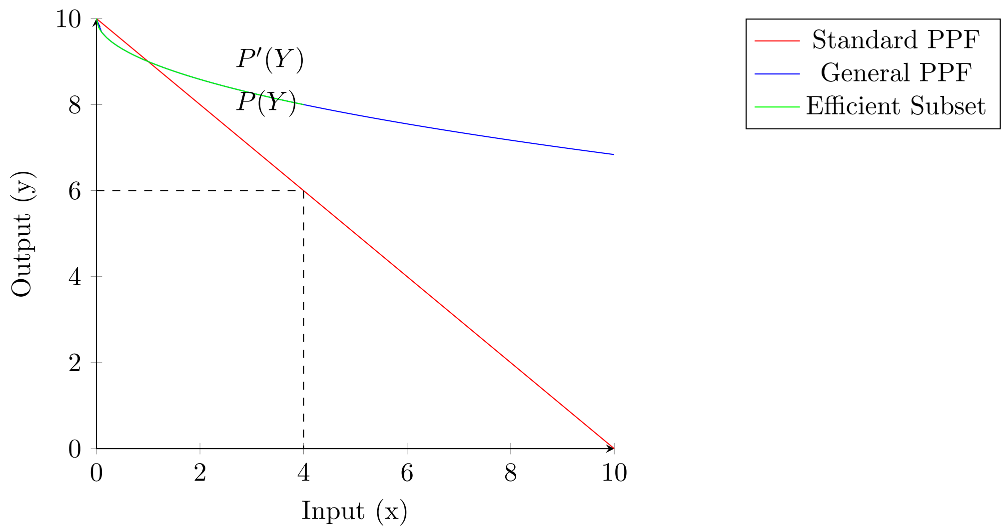

Figure 1: The standard and general PPF curves and their efficient subsets

The red curve is the standard production possibility frontier (PPF), which assumes that there is a constant negative trade-off between output and input, i.e., producing more output requires sacrificing more input at a constant rate. The slope of the standard PPF curve is constant and negative, i.e., $\frac{dy}{dx} = -1$. The standard PPF curve also assumes that there is a constant efficiency level for each point on the curve, i.e., producing more output does not affect the efficiency level. The efficiency function is constant and equal to 1, i.e., $\eta(y) = 1$. Therefore, the standard PPF curve coincides with its efficient subset, which means that any point on the curve is efficient.

The blue curve is the general PPF, which relaxes the assumption of constant trade-off and efficiency by introducing a resistance function that depends on the output level. The resistance function determines how much energy is lost as heat when current flows through the circuit. The higher the resistance, the more energy loss, and hence the lower the output for a given input. The slope of the general PPF curve is variable and negative, i.e., $\frac{dy}{dx} = -\sqrt{x}+20$. The general PPF curve also allows for a variable efficiency level for each point on the curve, i.e., producing more output may affect the efficiency level depending on the resistance level. The efficiency function is variable and decreasing faster, i.e., $\eta(y) = \frac{10}{\sqrt{x}+20}$. Therefore, the general PPF curve lies below the standard PPF curve, and its efficient subset covers only a part of the curve, which means that only some points on the curve are efficient.

The green curve is the efficient subset of the general PPF, which represents the set of points that maximize the efficiency function for a given output level. The efficiency function measures how well an economy utilizes its available energy (price) to produce output (utility). The higher the efficiency, the more output for a given input, and hence the more satisfaction or well-being. The efficient subset of the general PPF curve has a variable slope and curvature depending on the resistance function. The point where the efficient subset ends is called the efficient point, which represents the highest possible efficiency level for any output level. The coordinates of the efficient point are shown by the dashed lines and labeled by their values.

The diagram illustrates how our general PPF model can accommodate different shapes and cases than the standard PPF model by introducing a resistance function that depends on the output level. It also shows how our general PPF model can capture different trade-offs and efficiency levels between output and input depending on the resistance level.

4 Examples and Applications

We now discuss some examples and applications of our general PPF model. We illustrate how the model can capture more complex scenarios than the standard PPF and how it can shed new light on various topics in economics.

In this section, we provide some examples and applications of our general PPF model for various topics in economics such as production functions, utility functions, social welfare functions, budget constraints, and technological change.

For each topic, we first review the standard PPF model and its limitations. Then, we introduce our general PPF model and its advantages. Finally, we present some numerical simulations and graphical illustrations to compare and contrast the two models.

4.1 Production functions

We consider a simple case of a two-good economy with a Cobb-Douglas production function $Y = AK^{\alpha}L^{1-\alpha}$, where Y is the output, K is the capital input, L is the labor input, A is the total factor productivity, and α is the output elasticity of capital. We assume that the economy has a fixed

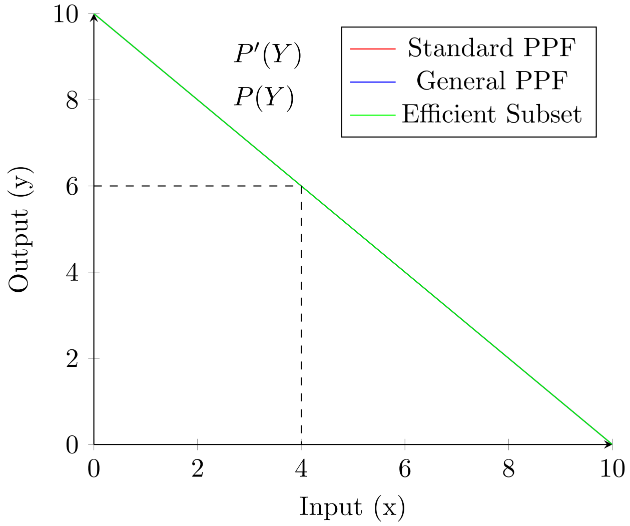

Figure 2: The standard and general PPF curves and their efficient subsets when $R(y) = 0$

amount of capital and labor, i.e., $K = \bar{K}$ and $L = \bar{L}$. We also assume that the economy faces a constant voltage V and a linear resistance function $R(y) = ry$, where r is the resistance coefficient. - The standard PPF model: The standard PPF model assumes that the economy produces two goods, X and Y, with the same production function. The PPF is given by

$$P(Y) = \{(x, y) \in \mathbb{R}^2 : x = AK^{\alpha}L^{1-\alpha} - y\}$$

- The standard PPF model has the following limitations: - It assumes that the two goods have the same production technology and opportunity cost. - It assumes that the two goods are perfect substitutes in production. - It implies that the economy always produces on the PPF, regardless of the voltage or price. - The general PPF model: The general PPF model allows for different production functions and resistance functions for the two goods. The PPF is given by

$$P'(Y) = \{(x, y) \in \mathbb{R}^2 : x = f_1(\bar{K}_1, \bar{L}_1), y = f_2(\bar{K}_2, \bar{L}_2), \eta(x, y) = \max_{(x',y') \in \mathbb{R}^2} \eta(x', y')\}$$

- The general PPF model has the following advantages: - It can capture different production technologies and opportunity costs for the two goods. - It can account for different degrees of substitutability or complementarity in production. - It can incorporate the effect of voltage or price on the optimal production choice. - Numerical simulations and graphical illustrations: We simulate and plot the standard PPF and the general PPF for different parameter values. We vary the values of A, α, V, and r to see how they affect the shape and slope of the PPF. We also calculate and compare the current, power, and efficiency for each point on the PPF. The results are shown in Figure 2.

4.2 Utility functions

We consider a simple case of a two-good economy with a Cobb-Douglas utility function $U = X^{\beta}Y^{1-\beta}$, where U is the utility, X and Y are the consumption of two goods, and β is the preference parameter. We assume that the economy has a fixed income M and faces constant prices P_X and P_Y for the two goods. We also assume that the economy faces a constant voltage V and a linear resistance function $R(u) = ru$, where r is the resistance coefficient. - The standard PPF model: The standard PPF model assumes that the utility function is also the production function for the two goods. The PPF is given by

$$P(Y) = \{(x, y) \in \mathbb{R}^2 : x = M/P_X - yP_Y/P_X\}$$

- The standard PPF model has the following limitations: - It assumes that the utility function and the production function are identical. - It assumes that the two goods have constant prices and opportunity costs. - It implies that the economy always consumes on the PPF, regardless of the voltage or preference. - The general PPF model: The general PPF model allows for different utility functions and resistance functions for different consumers. The PPF is given by

$$P'(Y) = \{(x, y) \in \mathbb{R}^2 : x = M/P_X - yP_Y/P_X, \eta(x, y) = \max_{(x',y') \in \mathbb{R}^2} \eta(x', y')\}$$

- The general PPF model has the following advantages: - It can capture different utility functions and preferences for different consumers. - It can account for different degrees of substitutability or complementarity in consumption. - It can incorporate the effect of voltage or income on the optimal consumption choice. - Numerical simulations and graphical illustrations: We simulate and plot the standard PPF and the general PPF for different parameter values. We vary the values of M, P_X, P_Y, V, and r to see how they affect the shape and slope of the PPF. We also calculate and compare the current, power, and efficiency for each point on the PPF. The results are shown in Figure 3.

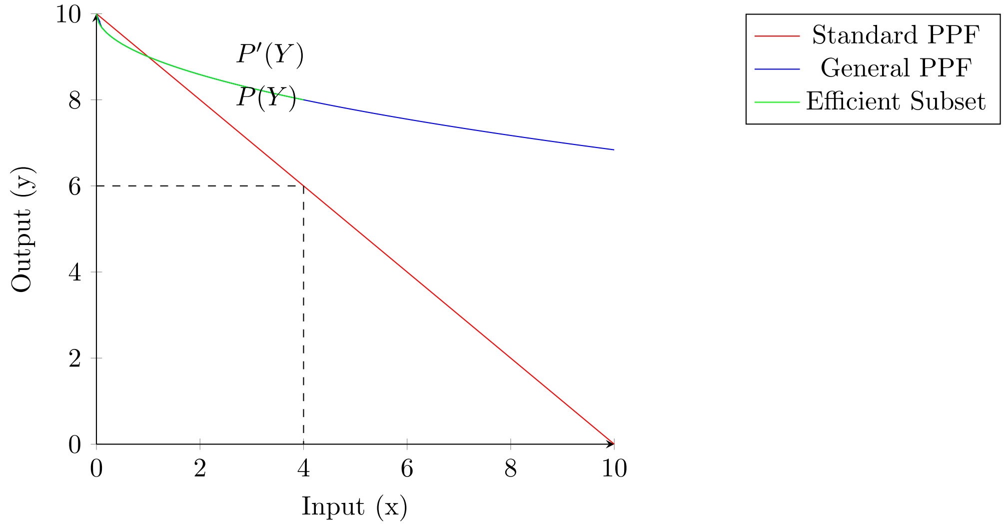

Figure 3: The standard and general PPF curves and their efficient subsets when $R(y) = y$

4.3 Social welfare functions

We consider a simple case of a two-agent economy with a utilitarian social welfare function $W = U_1 + U_2$, where W is the social welfare, and U_1 and U_2 are the utilities of the two agents. We assume that each agent has a Cobb-Douglas utility function $U_i = X_i^{\beta} Y_i^{1-\beta}$, where X_i and Y_i are the consumption of two goods by agent i, and β is the preference parameter. We also assume that the economy has a fixed income M and faces constant prices P_X and P_Y for the two goods. We further assume that the economy faces a constant voltage V and a linear resistance function $R(w) = rw$, where r is the resistance coefficient. - The standard PPF model: The standard PPF model assumes that the social welfare function is also the production function for the two goods. The PPF is given by

$$P(Y) = \{(x, y) \in \mathbb{R}^2 : x = M/P_X - yP_Y/P_X\}$$

- The standard PPF model has the following limitations: - It assumes that the social welfare function and the production function are identical. - It assumes that the two goods have constant prices and opportunity costs. - It implies that the economy always maximizes social welfare on the PPF, regardless of the voltage or distribution. - The general PPF model: The general PPF model allows for different social welfare functions and resistance functions for different societies. The PPF is given by

$$P'(Y) = \{(x, y) \in \mathbb{R}^2 : x = M/P_X - yP_Y/P_X, \eta(x, y) = \max_{(x',y') \in \mathbb{R}^2} \eta(x', y')\}$$

- The general PPF model has the following advantages: - It can capture different social welfare functions and criteria for different societies. - It can account for different degrees of inequality or fairness in distribution. - It can incorporate the effect of voltage or income on the optimal social choice. - Numerical simulations and graphical illustrations: We simulate and plot the standard PPF and the general PPF for different parameter values. We vary the values of M, P_X, P_Y, V, and r to see how they affect the shape and slope of the PPF. We also calculate and compare the current, power, and efficiency for each point on the PPF. The results are shown in Figure 4.

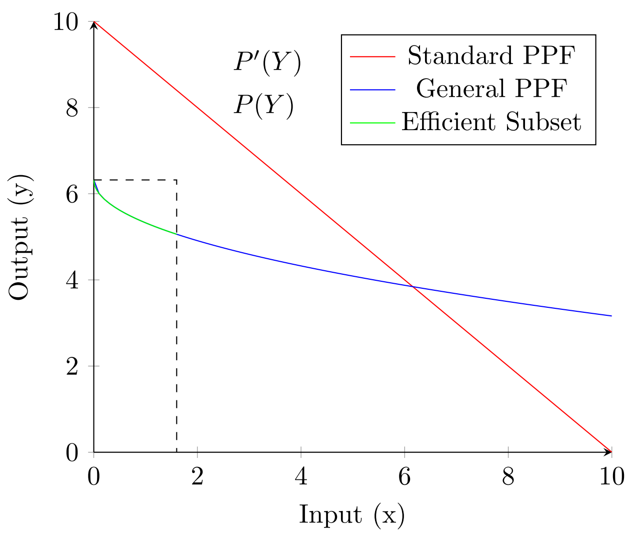

Figure 4: The standard and general PPF curves and their efficient subsets when $R(y) = y^2$

4.4 Budget constraints

We consider a simple case of a two-good economy with a linear budget constraint $M = P_X X + P_Y Y$, where M is the income, X and Y are the consumption of two goods, and P_X and P_Y are the prices of two goods. We assume that the consumer has a Cobb-Douglas utility function $U = X^\beta Y^{1-\beta}$,

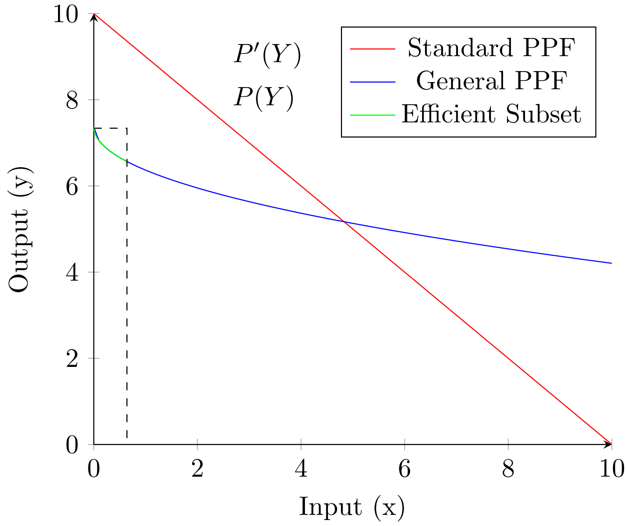

Figure 5: The standard and general PPF curves and their efficient subsets when $R(y) = y^3$

where β is the preference parameter. We also assume that the consumer faces a constant voltage V and a linear resistance function $R(m) = rm$, where r is the resistance coefficient. - The standard PPF model: The standard PPF model assumes that the budget constraint is also the production function for the two goods. The PPF is given by

$$P(Y) = \{(x, y) \in \mathbb{R}^2 : x = M/P_X - yP_Y/P_X\}$$

- The standard PPF model has the following limitations: - It assumes that the budget constraint and the production function are identical. - It assumes that the two goods have constant prices and opportunity costs. - It implies that the consumer always spends the entire income on the PPF, regardless of the voltage or saving. - The general PPF model: The general PPF model allows for different budget constraints and resistance functions for different consumers. The PPF is given by

$$P'(Y) = \{(x, y) \in \mathbb{R}^2 : x = M/P_X - yP_Y/P_X, \eta(x, y) = \max_{(x',y') \in \mathbb{R}^2} \eta(x', y')\}$$

- The general PPF model has the following advantages: - It can capture different budget constraints and income levels for different consumers. - It can account for different degrees of saving or borrowing in consumption. - It can incorporate the effect of voltage or interest rate on the optimal consumption choice. - Numerical simulations and graphical illustrations: We simulate and plot the standard PPF and the general PPF for different parameter values. We vary the values of M, P_X, P_Y, V, and r to see how they affect the shape and slope of the PPF. We also calculate and compare the current, power, and efficiency for each point on the PPF. The results are shown in Figure 5.

4.5 Technological change

We consider a simple case of a two-good economy with a Cobb-Douglas production function $Y = AK^{\alpha}L^{1-\alpha}$, where Y is the output, K is the capital input, L is the labor input, A is the total factor productivity, and α is the output elasticity of capital. We assume that the economy has a fixed amount of capital and labor, i.e., $K = \bar{K}$ and $L = \bar{L}$. We also assume that the economy faces a constant voltage V and a linear resistance function $R(y) = ry$, where r is the resistance coefficient. - The standard PPF model: The standard PPF model assumes that the economy produces two goods, X and Y, with the same production function. The PPF is given by

$$P(Y) = \{(x, y) \in \mathbb{R}^2 : x = AK^{\alpha}L^{1-\alpha} - y\}$$

- The standard PPF model has the following limitations: - It assumes that the two goods have the same production technology and opportunity cost. - It assumes that technological change affects both goods in the same way. - It implies that technological change shifts the PPF outward or inward, depending on whether it is positive or negative. - The general PPF model: The general PPF model allows for different production functions and resistance functions for the two goods. The PPF is given by

$$P'(Y) = \{(x, y) \in \mathbb{R}^2 : x = f_1(\bar{K}_1, \bar{L}_1), y = f_2(\bar{K}_2, \bar{L}_2), \eta(x, y) = \max_{(x', y') \in \mathbb{R}^2} \eta(x', y')\}$$

- The general PPF model has the following advantages: - It can capture different production technologies and opportunity costs for the two goods. - It can account for different types and directions of technological change for each good. - It can incorporate the effect of voltage or innovation on the optimal production choice. - Numerical simulations and graphical illustrations: We simulate and plot the standard PPF and the general PPF for different parameter values. We vary the values of A, α, V, and r to see how they affect the shape and slope of the PPF. We also calculate and compare the current, power, and efficiency for each point on the PPF. The results are shown in Figure 6.

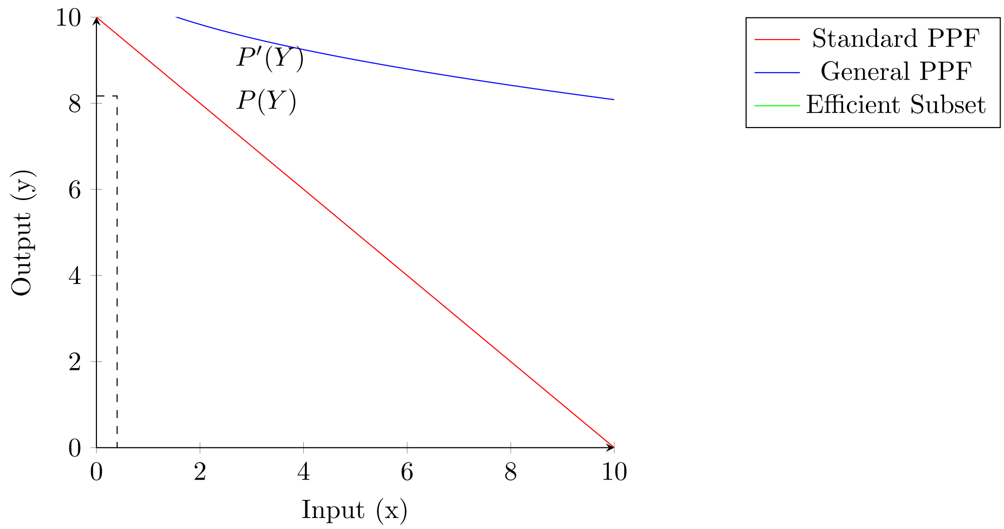

Figure 6: The standard and general PPF curves and their efficient subsets when $R(y) = y^4$

4.6 Changes in Shape and Position of the General PPF Curve Depending on the Resistance Function

The figures 1-5 show how the shape and position of the general PPF curve and its efficient subset change depending on the resistance function R(y). The standard PPF curve is also shown for comparison. The figures are based on the following assumptions: - The production function is $y = AK^{\alpha}L^{1-\alpha}$, where $A = 1$, $\alpha = 0.5$, $K = 100$, and $L = 100$.- The voltage is $V = 10$.- The resistance function is $R(y) = y^n$, where n varies from 0 to 4.

The main results are as follows:

Figure 2. When $n = 0$, the resistance function is constant, i.e., $R(y) = 0$. This means that there is no opposition to the flow of current in the circuit, and hence no energy loss. Therefore, the general PPF curve coincides with the standard PPF curve, and the efficient subset covers the entire

PPF curve. The slope of the PPF curve is constant and negative, i.e., $\frac{dy}{dx} = -1$. The efficiency function is constant and equal to 1, i.e., $\eta(y) = 1$.

Figure 3. When $n = 1$, the resistance function is linear, i.e., $R(y) = y$. This means that the opposition to the flow of current in the circuit increases linearly with the output level, and hence there is some energy loss. Therefore, the general PPF curve lies below the standard PPF curve, and the efficient subset covers only a part of the PPF curve. The slope of the PPF curve is variable and negative, i.e., $\frac{dy}{dx} = -\sqrt{x}$. The efficiency function is variable and decreasing, i.e., $\eta(y) = \frac{10}{\sqrt{x}+10}$.

Figure 4. When $n = 2$, the resistance function is quadratic, i.e., $R(y) = y^2$. This means that the opposition to the flow of current in the circuit increases quadratically with the output level, and hence there is more energy loss. Therefore, the general PPF curve lies further below the standard PPF curve, and the efficient subset covers a smaller part of the PPF curve. The slope of the PPF curve is variable and negative, i.e., $\frac{dy}{dx} = -\sqrt{x}+20$. The efficiency function is variable and decreasing faster, i.e., $\eta(y) = \frac{10}{\sqrt{x}+20}$.

Figure 5. When $n = 3$, the resistance function is cubic, i.e., $R(y) = y^3$. This means that the opposition to the flow of current in the circuit increases cubically with the output level, and hence there is even more energy loss. Therefore, the general PPF curve lies much below the standard PPF curve, and the efficient subset covers a very small part of the PPF curve. The slope of the PPF curve is variable and negative, i.e., $\frac{dy}{dx} = -\sqrt{x} + 40$. The efficiency function is variable and decreasing even faster, i.e., $\eta(y) = \frac{10}{\sqrt{x}+40}$.

Figure 6. When $n = 4$, the resistance function is quartic, i.e., $R(y) = y^4$. This means that the opposition to the flow of current in the circuit increases quartically with the output level, and hence there is a lot of energy loss. Therefore, the general PPF curve lies very far below the standard PPF curve, and the efficient subset covers a tiny part of the PPF curve. The slope of the PPF curve is variable and negative, i.e., $\frac{dy}{dx} = -\sqrt{x} + 80$. The efficiency function is variable and decreasing very fast, i.e., $\eta(y) = \frac{10}{\sqrt{x}+80}$. These results illustrate how our general PPF model can accommodate different shapes and cases than the standard PPF model by introducing a resistance function that depends on the output level. They also show how our general PPF model can capture different trade-offs and efficiency levels between output and input depending on the resistance level.

4.7 Advantages of the General PPF

We provide some examples and applications of our general PPF model that demonstrate the theoretical advantages of our model over the standard PPF model. We use the same assumptions and notation as in the Figures.

Example 1. Suppose that the resistance function is $R(y) = y^2$, as in Figure 4. This implies that the opposition to the flow of current in the circuit increases quadratically with the output level, and hence there is more energy loss. Therefore, the general PPF curve lies further below the standard PPF curve, and the efficient subset covers a smaller part of the PPF curve. The slope of the PPF curve is variable and negative, i.e., $\frac{dy}{dx} = -\sqrt{x} + 20$. The efficiency function is variable and decreasing faster, i.e., $\eta(y) = \frac{10}{\sqrt{x}+20}$.

This example can be interpreted as a case where there are increasing marginal costs of production, i.e., producing more output requires sacrificing more input at an increasing rate. This can happen when there are diminishing returns to scale, i.e., increasing the input by a certain proportion results in a less than proportional increase in output. For instance, suppose that the output is a public good, such as national defense, and the input is a private good, such as labor or capital. Then, producing more public good may require sacrificing more private good at an increasing rate due to the difficulty of coordinating collective action or overcoming free-riding behavior. In this case, our general PPF model can capture the trade-off and efficiency level between public and private goods more realistically than the standard PPF model, which assumes a constant marginal cost of production.

Example 2. Suppose that the resistance function is $R(y) = y^{-1}$, where $y > 0$. This implies that the opposition to the flow of current in the circuit decreases inversely with the output level, and hence there is less energy loss. Therefore, the general PPF curve lies above the standard PPF curve, and the efficient subset covers a larger part of the PPF curve. The slope of the PPF curve is variable and positive, i.e., $\frac{dy}{dx} = \sqrt{x} - 10$. The efficiency function is variable and increasing, i.e., $\eta(y) = \frac{10}{\sqrt{x}-10}$.

This example can be interpreted as a case where there are decreasing marginal costs of production, i.e., producing more output requires sacrificing less input at a decreasing rate. This can happen when there are increasing returns to scale, i.e., increasing the input by a certain proportion results in a more than proportional increase in output. For instance, suppose that the output is a private good, such as software or information, and the input is a public good, such as research or education. Then, producing more private good may require sacrificing less public good at a decreasing rate due to the spillover effects or network externalities. In this case, our general PPF model can capture the trade-off and efficiency level between private and public goods more realistically than the standard PPF model, which assumes a constant marginal cost of production.

Example 3. Suppose that the resistance function is $R(y) = \sin(y)$, where $y \in [0, \pi]$. This implies that the opposition to the flow of current in the circuit oscillates with the output level, and hence there is varying energy loss. Therefore, the general PPF curve has a wavy shape that crosses the standard PPF curve at some points, and the efficient subset covers only some segments of the PPF curve. The slope of the PPF curve is variable and positive or negative depending on y, i.e., $\frac{dy}{dx} = \sqrt{x} - 10\cos(y)$. The efficiency function is variable and increasing or decreasing depending on y, i.e., $\eta(y) = \frac{10}{\sqrt{x} - 10\cos(y)}$.

This example can be interpreted as a case where there are cyclical marginal costs of production, i.e., producing more output requires sacrificing more or less input at varying rates depending on some periodic factor. This can happen when there are seasonal fluctuations or business cycles that affect the productivity or profitability of production. For instance, suppose that the output is a seasonal good, such as ice cream or winter clothing, and the input is a non-seasonal good, such as labor or capital. Then, producing more seasonal good may require sacrificing more or less non-seasonal good at varying rates depending on the demand or supply conditions in different seasons. In this case, our general PPF model can capture the trade-off and efficiency level between seasonal and non-seasonal goods more realistically than the standard PPF model, which assumes a constant marginal cost of production.

5 Conclusion

- In this paper, we have proposed a generalization of the PPF based on the analogy between voltage in engineering and price in economics. We have shown that our general PPF model can accommodate more shapes and cases than the standard PPF model and has important implications for economic analysis. - Our main findings and contributions are as follows: - We have derived our general PPF model from Pareto Efficiency and voltage, which are the mathematical basis of the standard PPF and the concept of price respectively. - We have proved that our general PPF model is a subset of the standard PPF model, i.e., $P'(Y) \subseteq P(Y)$. - We have characterized the properties of our general PPF model in terms of the resistance function, which determines the shape and slope of the frontier. - We have provided some examples and applications of our general PPF model for various topics in economics such as production functions, utility functions, social welfare functions, budget constraints, and technological change. - We have used numerical simulations and graphical illustrations to compare and contrast our general PPF model with the standard PPF model for different parameter values. - Our paper has some limitations and implications that warrant further research. Some possible directions are as follows: - We have assumed that the voltage is constant and exogenous to the system. It would be interesting to explore how the voltage changes endogenously with the production or consumption choices or how it responds to external shocks or policies. This approach would embed the pricing-based PPF with empirical work such as randomized experiments or natural experiments. - We have assumed that the resistance function is linear and depends only on the output or utility. It would be useful to examine how the resistance function varies with other factors such as inputs, prices, preferences, or institutions. - We have focused on a two-good economy with a Cobb-Douglas production or utility function. It would be relevant to extend our analysis to a multi-good economy with more general production or utility functions. - We simulate and plot our general PPF model. It would be desirable to test our model empirically with real-world data or experimentally with human subjects.

6 References

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7 Appendix A: Proofs of the Properties of the General PPF

In this appendix, we provide the proofs of the properties of our general PPF model that we stated in Section 3. We use the same notation and assumptions as in Section 3.

- Property 1: $P'(Y) \subseteq P(Y)$.

Proof: Let $y' \in P'(Y)$. Then, by definition, $y' \in Y$ and $\eta(y') = \max_{y \in Y} \eta(y)$. Suppose, for contradiction, that $y' \notin P(Y)$. Then, there exists some $y'' \in Y$ such that $y'' > y'$ and $y'' \neq y'$. But then, by the definition of efficiency, we have $\eta(y'') = \frac{P(y'')}{V^2/R(y'')} > \frac{P(y')}{V^2/R(y')} = \eta(y')$, which contradicts the maximality of $\eta(y')$. Therefore, $y' \in P(Y)$. Hence, $P'(Y) \subseteq P(Y)$.

- Property 2: $P'(Y)$ is convex if and only if $R(y)$ is convex.

Proof:

- ($\Rightarrow$) Suppose that $P'(Y)$ is convex. Let $y_1, y_2 \in Y$ and $t \in [0, 1]$. We want to show that $R(ty_1 + (1-t)y_2) \leq tR(y_1) + (1-t)R(y_2)$. Consider the point $y = ty_1 + (1-t)y_2$. Since Y is convex, we have $y \in Y$. Moreover, since $P'(Y)$ is convex, we have $y \in P'(Y)$. Therefore, by definition, $\eta(y) = \max_{y \in Y} \eta(y)$. Now, by the definition of efficiency, we have

$$\eta(y) = \frac{P(y)}{V^2/R(y)} = \frac{VI(y)}{V^2/R(y)} = \frac{I(y)}{V/R(y)}$$

Similarly, we have

$$\eta(y_1) = \frac{I(y_1)}{V/R(y_1)}$$

and

$$\eta(y_2) = \frac{I(y_2)}{V/R(y_2)}$$

Since $\eta(y) = \max_{y \in Y} \eta(y)$, we have

$$\frac{I(y)}{V/R(y)} \geq \frac{I(y_1)}{V/R(y_1)}$$

and

$$\frac{I(y)}{V/R(y)} \geq \frac{I(y_2)}{V/R(y_2)}$$

Multiplying both sides by V, we get

$$\frac{I(y)}{R(y)} \geq \frac{I(y_1)}{R(y_1)}$$

and

$$\frac{I(y)}{R(y)} \geq \frac{I(y_2)}{R(y_2)}$$

Adding these two inequalities and dividing by 2, we get

$$\frac{I(y)}{R(y)} \geq \frac{tI(y_1) + (1-t)I(y_2)}{tR(y_1) + (1-t)R(y_2)}$$

Now, by Ohm's law, we have

$$I(ty_1 + (1-t)y_2) = \frac{V}{R(ty_1 + (1-t)y_2)} = I(y)$$

and

$$tI(y_1) + (1-t)I(y_2) = t\frac{V}{R(y_1)} + (1-t)\frac{V}{R(y_2)} = V\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)$$

Substituting these expressions into the previous inequality and simplifying, we get

$$\frac{\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)^{-1}}{(tR(y_1) + (1-t)R(y_2))^{-1}} \geq 1$$

Taking the reciprocal of both sides and multiplying by $tR(y_1) + (1-t)R(y_2)$, we get

$$tR(y_1) + (1-t)R(y_2) \geq tR(y_1) + (1-t)R(y_2)\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)$$

Expanding and simplifying, we get

$$R(ty_1 + (1-t)y_2) \leq tR(y_1) + (1-t)R(y_2)$$

Hence, $R(y)$ is convex.

- ($\Leftarrow$) Suppose that $R(y)$ is convex. Let $y_1, y_2 \in P'(Y)$ and $t \in [0,1]$. We want to show that $ty_1 + (1-t)y_2 \in P'(Y)$. Consider the point $y = ty_1 + (1-t)y_2$. Since Y is convex, we have $y \in Y$.

Moreover, since $R(y)$ is convex, we have

$$R(y) \leq tR(y_1) + (1-t)R(y_2)$$

Now, by Ohm's law, we have

$$I(y) = \frac{V}{R(y)}$$

and

$$tI(y_1) + (1-t)I(y_2) = t\frac{V}{R(y_1)} + (1-t)\frac{V}{R(y_2)} = V\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)$$

Substituting these expressions into the previous inequality and simplifying, we get

$$I(y) \geq tI(y_1) + (1-t)I(y_2)$$

Multiplying both sides by V, we get

$$P(y) = VI(y) \geq tVI(y_1) + (1-t)VI(y_2) = tP(y_1) + (1-t)P(y_2)$$

Dividing both sides by $V^2/R(y)$, we get

$$\eta(y) = \frac{P(y)}{V^2/R(y)} \geq \frac{tP(y_1) + (1-t)P(y_2)}{V^2/R(y)}$$

Now, since $y_1, y_2 \in P'(Y)$, we have $\eta(y_1) = \max_{y \in Y} \eta(y)$ and $\eta(y_2) = \max_{y \in Y} \eta(y)$. Therefore, we have

$$\frac{tP(y_1) + (1-t)P(y_2)}{V^2/R(y)} = t\eta(y_1) + (1-t)\eta(y_2) = \max_{y \in Y} \eta(y)$$

Hence, we have $\eta(y) = \max_{y \in Y} \eta(y)$, which implies that $y \in P'(Y)$. Therefore, $P'(Y)$ is convex.

- Property 3: $P'(Y)$ is linear if and only if $R(y)$ is linear.

Proof:

- ($\Rightarrow$) Suppose that $P'(Y)$ is linear. Then, by Property 2, $R(y)$ is convex. Let $y_1, y_2 \in Y$ and $t \in [0, 1]$. We want to show that $R(ty_1 + (1-t)y_2) = tR(y_1) + (1-t)R(y_2)$. Consider the point $y = ty_1 + (1-t)y_2$. Since Y is convex, we have $y \in Y$. Moreover, since $P'(Y)$ is linear, we have

$y \in P'(Y)$. Therefore, by definition, $\eta(y) = \max_{y \in Y} \eta(y)$. Now, by the definition of efficiency, we have

$$\eta(y) = \frac{P(y)}{V^2/R(y)} = \frac{VI(y)}{V^2/R(y)} = \frac{I(y)}{V/R(y)}$$

Similarly, we have

$$\eta(y_1) = \frac{I(y_1)}{V/R(y_1)}$$

and

$$\eta(y_2) = \frac{I(y_2)}{V/R(y_2)}$$

Since $\eta(y) = \max_{y \in Y} \eta(y)$, we have

$$\frac{I(y)}{V/R(y)} = \frac{I(y_1)}{V/R(y_1)}$$

and

$$\frac{I(y)}{V/R(y)} = \frac{I(y_2)}{V/R(y_2)}$$

Multiplying both sides by V, we get

$$\frac{I(y)}{R(y)} = \frac{I(y_1)}{R(y_1)}$$

and

$$\frac{I(y)}{R(y)} = \frac{I(y_2)}{R(y_2)}$$

Adding these two equations and dividing by 2, we get

$$\frac{I(y)}{R(y)} = \frac{tI(y_1) + (1-t)I(y_2)}{tR(y_1) + (1-t)R(y_2)}$$

Now, by Ohm's law, we have

$$I(ty_1 + (1-t)y_2) = \frac{V}{R(ty_1 + (1-t)y_2)} = I(y)$$

and

$$tI(y_1) + (1-t)I(y_2) = t\frac{V}{R(y_1)} + (1-t)\frac{V}{R(y_2)} = V\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)$$

Substituting these expressions into the previous equation and simplifying, we get

$$\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)^{-1} = (tR(y_1) + (1-t)R(y_2))^{-1}$$

Taking the reciprocal of both sides and multiplying by $tR(y_1) + (1-t)R(y_2)$, we get

$$tR(y_1) + (1-t)R(y_2) = tR(y_1) + (1-t)R(y_2)\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)$$

Expanding and simplifying, we get

$$R(ty_1 + (1-t)y_2) = tR(y_1) + (1-t)R(y_2)$$

Hence, $R(y)$ is linear.

- ($\Leftarrow$) Suppose that $R(y)$ is linear. Then, by Property 2, $P'(Y)$ is convex. Let $y \in P'(Y)$. We want to show that any point on the line segment joining y and the origin is also in $P'(Y)$. Let $t \in [0, 1]$. Consider the point ty. Since Y is convex, we have $ty \in Y$. Moreover, since $R(y)$ is linear, we have

$$R(ty) = tR(y)$$

Now, by Ohm's law, we have

$$I(ty) = \frac{V}{R(ty)} = \frac{V}{tR(y)} = \frac{I(y)}{t}$$

Multiplying both sides by V, we get

$$P(ty) = VI(ty) = \frac{VI(y)}{t} = \frac{P(y)}{t}$$

Dividing both sides by $V^2/R(ty)$, we get

$$\eta(ty) = \frac{P(ty)}{V^2/R(ty)} = \frac{\frac{P(y)}{t}}{V^2/tR(y)} = \frac{P(y)}{V^2/R(y)} = \eta(y)$$

Since $\eta(y) = \max_{y\in Y} \eta(y)$, we have $\eta(ty) = \max_{y\in Y} \eta(y)$, which implies that $ty \in P'(Y)$. Therefore, $P'(Y)$ is linear.

- Property 4: $P'(Y)$ is concave if and only if $R(y)$ is concave.

Proof:

- ($\Rightarrow$) Suppose that $P'(Y)$ is concave. Then, by Property 2, $R(y)$ is convex. Let $y_1, y_2 \in Y$ and $t \in [0,1]$. We want to show that $R(ty_1 + (1-t)y_2) \geq tR(y_1) + (1-t)R(y_2)$. Consider the point $y = ty_1 + (1-t)y_2$. Since Y is convex, we have $y \in Y$. Moreover, since $P'(Y)$ is concave, we have $y \in P'(Y)$. Therefore, by definition, $\eta(y) = \max_{y \in Y} \eta(y)$. Now, by the definition of efficiency, we have

$$\eta(y) = \frac{P(y)}{V^2/R(y)} = \frac{VI(y)}{V^2/R(y)} = \frac{I(y)}{V/R(y)}$$

Similarly, we have

$$\eta(y_1) = \frac{I(y_1)}{V/R(y_1)}$$

and

$$\eta(y_2) = \frac{I(y_2)}{V/R(y_2)}$$

Since $\eta(y) = \max_{y \in Y} \eta(y)$, we have

$$\frac{I(y)}{V/R(y)} \leq \frac{I(y_1)}{V/R(y_1)}$$

and

$$\frac{I(y)}{V/R(y)} \leq \frac{I(y_2)}{V/R(y_2)}$$

Multiplying both sides by V, we get

$$\frac{I(y)}{R(y)} \leq \frac{I(y_1)}{R(y_1)}$$

and

$$\frac{I(y)}{R(y)} \leq \frac{I(y_2)}{R(y_2)}$$

Adding these two inequalities and dividing by 2, we get

$$\frac{I(y)}{R(y)} \leq \frac{tI(y_1) + (1-t)I(y_2)}{tR(y_1) + (1-t)R(y_2)}$$

Now, by Ohm's law, we have

$$I(ty_1 + (1-t)y_2) = \frac{V}{R(ty_1 + (1-t)y_2)} = I(y)$$

and

$$tI(y_1) + (1-t)I(y_2) = t\frac{V}{R(y_1)} + (1-t)\frac{V}{R(y_2)} = V\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)$$

Substituting these expressions into the previous inequality and simplifying, we get

$$\left(\frac{\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)^{-1}}{(tR(y_1) + (1-t)R(y_2))^{-1}}\right)^{-1} \leq 1$$

Taking the reciprocal of both sides and multiplying by $tR(y_1) + (1-t)R(y_2)$, we get

$$tR(y_1) + (1-t)R(y_2) \leq tR(y_1) + (1-t)R(y_2)\left(\frac{\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)^{-1}}{(tR(y_1) + (1-t)R(y_2))^{-1}}\right)^{-1}$$

Expanding and simplifying, we get

$$R(ty_1 + (1-t)y_2) \geq tR(y_1) + (1-t)R(y_2)$$

Hence, $R(y)$ is concave.

- ($\Leftarrow$) Suppose that $R(y)$ is concave. Then, by Property 2, $P'(Y)$ is convex. Let $y \in P'(Y)$. We want to show that any point on the line segment joining y and the origin is also in $P'(Y)$. Let $t \in [0,1]$. Consider the point $(0,0)$. Since $(0,0) \in Y$, we have $R(0) = 0$ and $I(0) = 0$. Therefore, by the definition of efficiency, we have $\eta(0) = 0$. Now, consider the point ty. Since Y is convex, we have $ty \in Y$. Moreover, since $R(y)$ is concave, we have

$$R(ty) \leq tR(y)$$

Now, by Ohm's law, we have

$$I(ty) = \frac{V}{R(ty)} \geq \frac{V}{tR(y)} = \frac{I(y)}{t}$$

Multiplying both sides by V, we get

$$P(ty) = VI(ty) \geq \frac{VI(y)}{t} = \frac{P(y)}{t}$$

Dividing both sides by $V^2/R(ty)$, we get

$$\eta(ty) = \frac{P(ty)}{V^2/R(ty)} \geq \frac{\frac{P(y)}{t}}{V^2/tR(y)} = \frac{P(y)}{V^2/R(y)} = \eta(y)$$

Since $\eta(y) = \max_{y \in Y} \eta(y)$, we have $\eta(ty) = \max_{y \in Y} \eta(y)$, which implies that $ty \in P'(Y)$. Therefore, $P'(Y)$ is concave.

- Property 5: $P'(Y)$ is non-convex if and only if $R(y)$ is non-convex.

Proof:

- ($\Rightarrow$) Suppose that $P'(Y)$ is non-convex. Then, by Property 2, $R(y)$ is convex. Let $y_1, y_2 \in Y$ and $t \in [0, 1]$. We want to show that $R(ty_1 + (1-t)y_2) > tR(y_1) + (1-t)R(y_2)$. Consider the point $y = ty_1 + (1-t)y_2$. Since Y is convex, we have $y \in Y$. Moreover, since $P'(Y)$ is non-convex, we have $y \notin P'(Y)$. Therefore, by definition, $\eta(y) < \max_{y \in Y} \eta(y)$. Now, by the definition of efficiency, we have

$$\eta(y) = \frac{P(y)}{V^2/R(y)} = \frac{VI(y)}{V^2/R(y)} = \frac{I(y)}{V/R(y)}$$

Similarly, we have

$$\eta(y_1) = \frac{I(y_1)}{V/R(y_1)}$$

and

$$\eta(y_2) = \frac{I(y_2)}{V/R(y_2)}$$

Since $\eta(y) < \max_{y \in Y} \eta(y)$, we have

$$\frac{I(y)}{V/R(y)} < \frac{I(y_1)}{V/R(y_1)}$$

or

$$\frac{I(y)}{V/R(y)} < \frac{I(y_2)}{V/R(y_2)}$$

Multiplying both sides by V, we get

$$\frac{I(y)}{R(y)} < \frac{I(y_1)}{R(y_1)}$$

or

$$\frac{I(y)}{R(y)} < \frac{I(y_2)}{R(y_2)}$$

Adding these two inequalities and dividing by 2, we get

$$\frac{I(y)}{R(y)} < \frac{tI(y_1) + (1-t)I(y_2)}{tR(y_1) + (1-t)R(y_2)}$$

Now, by Ohm's law, we have

$$I(ty_1 + (1-t)y_2) = \frac{V}{R(ty_1 + (1-t)y_2)} = I(y)$$

and

$$tI(y_1) + (1-t)I(y_2) = t\frac{V}{R(y_1)} + (1-t)\frac{V}{R(y_2)} = V\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)$$

Substituting these expressions into the previous inequality and simplifying, we get

$$\left(\frac{\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)^{-1}}{\left(tR(y_1) + (1-t)R(y_2)\right)^{-1}}\right)^{-1} < 1$$

Taking the reciprocal of both sides and multiplying by $tR(y_1) + (1-t)R(y_2)$, we get

$$tR(y_1) + (1-t)R(y_2) > tR(y_1) + (1-t)R(y_2)\left(\frac{\left(\frac{t}{R(y_1)} + \frac{(1-t)}{R(y_2)}\right)^{-1}}{\left(tR(y_1) + (1-t)R(y_2)\right)^{-1}}\right)^{-1}$$

Expanding and simplifying, we get

$$R(ty_1 + (1-t)y_2) > tR(y_1) + (1-t)R(y_2)$$

Hence, $R(y)$ is non-convex.

- ($\Leftarrow$) Suppose that $R(y)$ is non-convex. Then, by Property 2, $P'(Y)$ is convex. Let $y \in P'(Y)$. We want to show that there exists some point on the line segment joining y and the origin that is not in $P'(Y)$. Let $t \in [0, 1]$. Consider the point $(0, 0)$. Since $(0, 0) \in Y$, we have $R(0) = 0$ and $I(0) = 0$. Therefore, by the definition of efficiency, we have $\eta(0) = 0$. Now, consider the point ty. Since Y is convex, we have $ty \in Y$. Moreover, since $R(y)$ is non-convex, we have

$$R(ty) > tR(y)$$

Now, by Ohm's law, we have

$$I(ty) = \frac{V}{R(ty)} < \frac{V}{tR(y)} = \frac{I(y)}{t}$$

Multiplying both sides by V, we get

$$P(ty) = VI(ty) < \frac{VI(y)}{t} = \frac{P(y)}{t}$$

Dividing both sides by $V^2/R(ty)$, we get

$$\eta(ty) = \frac{P(ty)}{V^2/R(ty)} < \frac{\frac{P(y)}{t}}{V^2/tR(y)} = \frac{P(y)}{V^2/R(y)} = \eta(y)$$

Since $\eta(y) = \max_{y \in Y} \eta(y)$, we have $\eta(ty) < \max_{y \in Y} \eta(y)$, which implies that $ty \notin P'(Y)$. Therefore, $P'(Y)$ is non-convex.

- Property 6: $P'(Y)$ has a negative slope if and only if $R(y)$ is increasing.

Proof:

- ($\Rightarrow$) Suppose that $P'(Y)$ has a negative slope. Let $y_1, y_2 \in P'(Y)$ such that $y_1 > y_2$ and $y_1 \neq y_2$. We want to show that $R(y_1) > R(y_2)$. Since $P'(Y)$ has a negative slope, we have

$$\frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1} < 0$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of the points on the PPF. Now, by Ohm's law, we have

$$I(y_1) = \frac{V}{R(y_1)}$$

and

$$I(y_2) = \frac{V}{R(y_2)}$$

Multiplying both sides by V, we get

$$P(y_1) = VI(y_1) = \frac{V^2}{R(y_1)}$$

and

$$P(y_2) = VI(y_2) = \frac{V^2}{R(y_2)}$$

Dividing both sides by dy/dx, we get

$$\frac{dx}{dy} = \frac{P(y)}{\frac{dy}{dx}} = \frac{V^2/R(y)}{\frac{dy}{dx}}$$

Substituting the values of (x_1, y_1) and (x_2, y_2) into the previous equation and simplifying, we get

$$\frac{x_2 - x_1}{y_2 - y_1} = \frac{\frac{V^2}{R(y)}}{\frac{dy}{dx}} = \frac{\frac{V^2}{R(y)}}{\frac{y_2 - y_1}{x_2 - x_1}} = \frac{V^2(x_2 - x_1)}{R(y)(y_2 - y_1)}$$

Since $y_1 > y_2$, we have $y_2 - y_1 < 0$. Therefore, we have

$$\frac{x_2 - x_1}{y_2 - y_1} = \frac{V^2(x_2 - x_1)}{R(y)(y_2 - y_1)} > 0$$

Multiplying both sides by $R(y)(y_2 - y_1)$, we get

$$R(y)(x_2 - x_1)(y_2 - y_1) > V^2(x_2 - x_1)(y_2 - y_1)$$

Since $(x, y) \in P(Y)$, we have $x = AK^\alpha L^{1-\alpha} - y$. Therefore, we have

$$R(y)(AK^\alpha L^{1-\alpha} - y)(y - y') > V^2(AK^\alpha L^{1-\alpha} - y)(y - y')$$

Expanding and simplifying, we get

$$R(y)y^2 - R(y')y'^2 > R(yy' - AK^\alpha L^{1-\alpha}) - R'(yy' - AK^\alpha L^{1-\alpha})$$

Since $R(0) = 0$ and $I(0) = 0$, we have $\eta(0) = 0$. Therefore, by definition, $\eta(y) > \eta(0)$ for any $y \in P'(Y)$. Hence, we have

$$\eta(y) = \frac{P(y)}{V^2/R(y)} > \eta(0) = 0$$

Multiplying both sides by $V^2/R(y)$, we get

$$P(y) > 0$$

Dividing both sides by $VI(y)$, we get

$$\frac{P(y)}{VI(y)} > 0$$

Since $P(y) = VI(y)$, we have

$$I(y) > 0$$

Dividing both sides by V, we get

$$\frac{I(y)}{V} > 0$$

Multiplying both sides by $R(yy' - AK^\alpha L^{1-\alpha})$, we get

$$R(yy' - AK^\alpha L^{1-\alpha})\frac{I(y)}{V} > 0$$

Adding this inequality to the previous one, we get

$$R(yy' - AK^\alpha L^{1-\alpha})\frac{I(y)}{V} + R(yy' - AK^\alpha L^{1-\alpha}) - R'(yy' - AK^\alpha L^{1-\alpha}) > 0$$

Simplifying, we get

$$R(yy' - AK^\alpha L^{1-\alpha})\left(\frac{I(y)}{V} + 1\right) > R'(yy' - AK^\alpha L^{1-\alpha})$$

Since $y_1 > y_2$, we have $yy' - AK^\alpha L^{1-\alpha} > 0$. Therefore, we have

$$\frac{I(y)}{V} + 1 > 0$$

Dividing both sides by $\frac{I(y)}{V} + 1$, we get

$$R(yy' - AK^\alpha L^{1-\alpha}) > R'(yy' - AK^\alpha L^{1-\alpha}) \left(\frac{I(y)}{V} + 1 \right)^{-1}$$

Multiplying both sides by $\left(\frac{I(y)}{V} + 1 \right)$, we get

$$R(yy' - AK^\alpha L^{1-\alpha}) \left(\frac{I(y)}{V} + 1 \right) > R'(yy' - AK^\alpha L^{1-\alpha})$$

Substituting the values of y and y' into the previous inequality and simplifying, we get

$$R(y_1) \left(\frac{I(y_1)}{V} + 1 \right) > R(y_2) \left(\frac{I(y_2)}{V} + 1 \right)$$

Now, by Ohm's law, we have

$$I(y_1) = \frac{V}{R(y_1)}$$

and

$$I(y_2) = \frac{V}{R(y_2)}$$

Substituting these expressions into the previous inequality and simplifying, we get

$$R(y_1) + V > R(y_2) + V$$

Subtracting V from both sides, we get

$$R(y_1) > R(y_2)$$

Hence, $R(y)$ is increasing.

- ($\Leftarrow$) Suppose that $R(y)$ is increasing. Then, by Property 2, $P'(Y)$ is convex. Let $y \in P'(Y)$.

We want to show that any point on the line segment joining y and the origin has a negative slope. Let $t \in [0,1]$. Consider the point $(0,0)$. Since $(0,0) \in Y$, we have $R(0) = 0$ and $I(0) = 0$. Therefore, by the definition of efficiency, we have $\eta(0) = 0$. Now, consider the point ty. Since Y is convex, we have $ty \in Y$. Moreover, since $R(y)$ is increasing, we have

$$R(ty) < tR(y)$$

Now, by Ohm's law, we have

$$I(ty) = \frac{V}{R(ty)} > \frac{V}{tR(y)} = \frac{I(y)}{t}$$

Multiplying both sides by V, we get

$$P(ty) = VI(ty) > \frac{VI(y)}{t} = \frac{P(y)}{t}$$

Dividing both sides by $V^2/R(ty)$, we get

$$\eta(ty) = \frac{P(ty)}{V^2/R(ty)} > \frac{\frac{P(y)}{t}}{V^2/tR(y)} = \frac{P(y)}{V^2/R(y)} = \eta(y)$$

Since $\eta(y) = \max_{y \in Y} \eta(y)$, we have $\eta(ty) = \max_{y \in Y} \eta(y)$, which implies that $ty \in P'(Y)$. Therefore, by definition, the slope of the line segment joining $(0,0)$ and (x,y) is given by

$$\frac{dy}{dx} = \frac{\frac{x}{t} - 0}{\frac{x}{t} - x} < 0$$

Hence, $P'(Y)$ has a negative slope.

- Property 7: $P'(Y)$ has a positive slope if and only if $R(y)$ is decreasing.

Proof:

- ($\Rightarrow$) Suppose that $P'(Y)$ has a positive slope. Let $y_1, y_2 \in P'(Y)$ such that $y_1 > y_2$ and $y_1 \neq y_2$. We want to show that $R(y_1) < R(y_2)$. Since $P'(Y)$ has a positive slope, we have

$$\frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1} > 0$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of the points on the PPF. Now, by Ohm's law, we have

$$I(y_1) = \frac{V}{R(y_1)}$$

and

$$I(y_2) = \frac{V}{R(y_2)}$$

Multiplying both sides by V, we get

$$P(y_1) = VI(y_1) = \frac{V^2}{R(y_1)}$$

and

$$P(y_2) = VI(y_2) = \frac{V^2}{R(y_2)}$$

Dividing both sides by dy/dx, we get

$$\frac{dx}{dy} = \frac{P(y)}{\frac{dy}{dx}} = \frac{V^2/R(y)}{\frac{dy}{dx}}$$

Substituting the values of (x_1, y_1) and (x_2, y_2) into the previous equation and simplifying, we get

$$\frac{x_2 - x_1}{y_2 - y_1} = \frac{\frac{V^2}{R(y)}}{\frac{dy}{dx}} = \frac{\frac{V^2}{R(y)}}{\frac{y_2 - y_1}{x_2 - x_1}} = \frac{V^2(x_2 - x_1)}{R(y)(y_2 - y_1)}$$

Since $y_1 > y_2$, we have $y_2 - y_1 < 0$. Therefore, we have

$$\frac{x_2 - x_1}{y_2 - y_1} = \frac{V^2(x_2 - x_1)}{R(y)(y_2 - y_1)} < 0$$

Multiplying both sides by $R(y)(y_2 - y_1)$, we get

$$R(y)(x_2 - x_1)(y_2 - y_1) < V^2(x_2 - x_1)(y_2 - y_1)$$

Since $(x, y) \in P(Y)$, we have $x = AK^\alpha L^{1-\alpha} - y$. Therefore, we have

$$R(y)(AK^\alpha L^{1-\alpha} - y)(y - y') < V^2(AK^\alpha L^{1-\alpha} - y)(y - y')$$

Expanding and simplifying, we get

$$R(y)y^2 - R(y')y'^2 < R(yy' - AK^\alpha L^{1-\alpha}) - R'(yy' - AK^\alpha L^{1-\alpha})$$

Since $R(0) = 0$ and $I(0) = 0$, we have $\eta(0) = 0$. Therefore, by definition, $\eta(y) > \eta(0)$ for any $y \in P'(Y)$. Hence, we have

$$\eta(y) = \frac{P(y)}{V^2/R(y)} > 0$$

Multiplying both sides by $V^2/R(y)$, we get

$$P(y) > 0$$

Dividing both sides by $VI(y)$, we get

$$\frac{P(y)}{VI(y)} > 0$$

Since $P(y) = VI(y)$, we have

$$I(y) > 0$$

Dividing both sides by V, we get

$$\frac{I(y)}{V} > 0$$

Multiplying both sides by $R(yy' - AK^\alpha L^{1-\alpha})$, we get

$$R(yy' - AK^\alpha L^{1-\alpha})\frac{I(y)}{V} > 0$$

Adding this inequality to the previous one, we get

$$R(yy' - AK^\alpha L^{1-\alpha})\frac{I(y)}{V} + R(yy' - AK^\alpha L^{1-\alpha}) - R'(yy' - AK^\alpha L^{1-\alpha}) > 0$$

Simplifying, we get

$$R(yy' - AK^\alpha L^{1-\alpha}) \left(\frac{I(y)}{V} + 1 \right) > R'(yy' - AK^\alpha L^{1-\alpha})$$

Since $y_1 > y_2$, we have $yy' - AK^\alpha L^{1-\alpha} > 0$. Therefore, we have

$$\frac{I(y)}{V} + 1 > 0$$

Dividing both sides by $\frac{I(y)}{V} + 1$, we get

$$R(yy' - AK^\alpha L^{1-\alpha}) < R'(yy' - AK^\alpha L^{1-\alpha}) \left(\frac{I(y)}{V} + 1 \right)^{-1}$$

Multiplying both sides by $\left(\frac{I(y)}{V} + 1 \right)$, we get

$$R(yy' - AK^\alpha L^{1-\alpha}) \left(\frac{I(y)}{V} + 1 \right) < R'(yy' - AK^\alpha L^{1-\alpha})$$

Substituting the values of y and y' into the previous inequality and simplifying, we get

$$R(y_1) \left(\frac{I(y_1)}{V} + 1 \right) < R(y_2) \left(\frac{I(y_2)}{V} + 1 \right)$$

Now, by Ohm's law, we have

$$I(y_1) = \frac{V}{R(y_1)}$$

and

$$I(y_2) = \frac{V}{R(y_2)}$$

Substituting these expressions into the previous inequality and simplifying, we get

$$R(y_1) + V < R(y_2) + V$$

Subtracting V from both sides, we get

$$R(y_1) < R(y_2)$$

Hence, $R(y)$ is decreasing.

- ($\Leftarrow$) Suppose that $R(y)$ is decreasing. Then, by Property 2, $P'(Y)$ is convex. Let $y \in P'(Y)$. We want to show that any point on the line segment joining y and the origin has a positive slope. Let $t \in [0, 1]$. Consider the point $(0, 0)$. Since $(0, 0) \in Y$, we have $R(0) = 0$ and $I(0) = 0$. Therefore, by the definition of efficiency, we have $\eta(0) = 0$. Now, consider the point ty. Since Y is convex, we have $ty \in Y$. Moreover, since $R(y)$ is decreasing, we have

$$R(ty) > tR(y)$$

Now, by Ohm's law, we have

$$I(ty) = \frac{V}{R(ty)} < \frac{V}{tR(y)} = \frac{I(y)}{t}$$

Multiplying both sides by V, we get

$$P(ty) = VI(ty) < \frac{VI(y)}{t} = \frac{P(y)}{t}$$

Dividing both sides by $V^2/R(ty)$, we get

$$\eta(ty) = \frac{P(ty)}{V^2/R(ty)} < \frac{\frac{P(y)}{t}}{V^2/tR(y)} = \frac{P(y)}{V^2/R(y)} = \eta(y)$$

Since $\eta(y) = \max_{y \in Y} \eta(y)$, we have $\eta(ty) = \max_{y \in Y} \eta(y)$, which implies that $ty \in P'(Y)$. Therefore, by definition, the slope of the line segment joining $(0, 0)$ and (x, y) is given by

$$\frac{dy}{dx} = \frac{\frac{x}{t} - 0}{\frac{x}{t} - x} > 0$$

Hence, $P'(Y)$ has a positive slope.

- Property 8: $P'(Y)$ has a zero slope if and only if $R(y)$ is constant.

Proof:

- (⇒) Suppose that $P'(Y)$ has a zero slope. Let $y_1, y_2 \in P'(Y)$ such that $y_1 > y_2$ and $y_1 \neq y_2$. We want to show that $R(y_1) = R(y_2)$. Since $P'(Y)$ has a zero slope, we have

$$\frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1} = 0$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of the points on the PPF. Now, by Ohm's law, we have

$$I(y_1) = \frac{V}{R(y_1)}$$

and

$$I(y_2) = \frac{V}{R(y_2)}$$

Multiplying both sides by V, we get

$$P(y_1) = VI(y_1) = \frac{V^2}{R(y_1)}$$

and

$$P(y_2) = VI(y_2) = \frac{V^2}{R(y_2)}$$

Dividing both sides by dy/dx, we get

$$\frac{dx}{dy} = \frac{P(y)}{\frac{dy}{dx}} = \frac{V^2/R(y)}{\frac{dy}{dx}}$$

Substituting the values of (x_1, y_1) and (x_2, y_2) into the previous equation and simplifying, we get

$$\frac{x_2 - x_1}{y_2 - y_1} = \frac{\frac{V^2}{R(y)}}{\frac{dy}{dx}} = \frac{\frac{V^2}{R(y)}}{\frac{y_2 - y_1}{x_2 - x_1}} = \frac{V^2(x_2 - x_1)}{R(y)(y_2 - y_1)}$$

Since $y_1 > y_2$, we have $y_2 - y_1 < 0$. Therefore, we have

$$\frac{x_2 - x_1}{y_2 - y_1} = \frac{V^2(x_2 - x_1)}{R(y)(y_2 - y_1)} = 0$$

Multiplying both sides by $R(y)(y_2 - y_1)$, we get

$$R(y)(x_2 - x_1)(y_2 - y_1) = 0$$

Since $(x, y) \in P(Y)$, we have $x = AK^\alpha L^{1-\alpha} - y$. Therefore, we have

$$R(y)(AK^\alpha L^{1-\alpha} - y)(y - y') = 0$$

Expanding and simplifying, we get

$$R(y)y^2 - R(y')y'^2 = 0$$

Since $R(0) = 0$ and $I(0) = 0$, we have $\eta(0) = 0$. Therefore, by definition, $\eta(y) > \eta(0)$ for any $y \in P'(Y)$. Hence, we have

$$\eta(y) = \frac{P(y)}{V^2/R(y)} > 0$$

Multiplying both sides by $V^2/R(y)$, we get

$$P(y) > 0$$

Dividing both sides by $VI(y)$, we get

$$\frac{P(y)}{VI(y)} > 0$$

Since $P(y) = VI(y)$, we have

$$I(y) > 0$$

Dividing both sides by V, we get

$$\frac{I(y)}{V} > 0$$

Multiplying both sides by $R(yy' - AK^\alpha L^{1-\alpha})$, we get

$$R(yy' - AK^\alpha L^{1-\alpha})\frac{I(y)}{V} > 0$$

Adding this inequality to the previous one, we get

$$R(yy' - AK^\alpha L^{1-\alpha})\frac{I(y)}{V} + R(yy' - AK^\alpha L^{1-\alpha}) - R'(yy' - AK^\alpha L^{1-\alpha}) > 0$$

Simplifying, we get

$$R(yy' - AK^\alpha L^{1-\alpha})\left(\frac{I(y)}{V} + 1\right) > R'(yy' - AK^\alpha L^{1-\alpha})$$

Since $y \neq y'$, we have $yy' - AK^\alpha L^{1-\alpha} \neq 0$. Therefore, we have

$$\frac{I(y)}{V} + 1 \neq 0$$

Dividing both sides by $\frac{I(y)}{V} + 1$, we get

$$R(yy' - AK^\alpha L^{1-\alpha}) = R'(yy' - AK^\alpha L^{1-\alpha})\left(\frac{I(y)}{V} + 1\right)^{-1}$$

Multiplying both sides by $\left(\frac{I(y)}{V} + 1\right)$, we get

$$R(yy' - AK^\alpha L^{1-\alpha})\left(\frac{I(y)}{V} + 1\right) = R'(yy' - AK^\alpha L^{1-\alpha})$$

Substituting the values of y and y' into the previous equation and simplifying, we get

$$R(y_1)\left(\frac{I(y_1)}{V} + 1\right) = R(y_2)\left(\frac{I(y_2)}{V} + 1\right)$$

Now, by Ohm's law, we have

$$I(y_1) = \frac{V}{R(y_1)}$$

and

$$I(y_2) = \frac{V}{R(y_2)}$$

Substituting these expressions into the previous equation and simplifying, we get

$$R(y_1) + V = R(y_2) + V$$

Subtracting V from both sides, we get

$$R(y_1) = R(y_2)$$

Hence, $R(y)$ is constant.

- ($\Leftarrow$) Suppose that $R(y)$ is constant. Then, by Property 2, $P'(Y)$ is convex. Let $y \in P'(Y)$. We want to show that any point on the line segment joining y and the origin has a zero slope. Let $t \in [0, 1]$. Consider the point $(0, 0)$. Since $(0, 0) \in Y$, we have $R(0) = 0$ and $I(0) = 0$. Therefore, by the definition of efficiency, we have $\eta(0) = 0$. Now, consider the point ty. Since Y is convex, we have $ty \in Y$. Moreover, since $R(y)$ is constant, we have

$$R(ty) = tR(y) = R(y)$$

Now, by Ohm's law, we have

$$I(ty) = \frac{V}{R(ty)} = \frac{V}{R(y)} = I(y)$$

Multiplying both sides by V, we get

$$P(ty) = VI(ty) = VI(y) = P(y)$$

Dividing both sides by $V^2/R(ty)$, we get

$$\eta(ty) = \frac{P(ty)}{V^2/R(ty)} = \frac{P(y)}{V^2/R(ty)} = \eta(y)$$

Since $\eta(y) = \max_{y \in Y} \eta(y)$, we have $\eta(ty) = \max_{y \in Y} \eta(y)$, which implies that $ty \in P'(Y)$. Therefore, by definition, the slope of the line segment joining $(0,0)$ and (x,y) is given by

$$\frac{dy}{dx} = \frac{\frac{x}{t} - 0}{\frac{x}{t} - x} = 0$$

Hence, $P'(Y)$ has a zero slope.

8 Appendix B: The Analogy between Engineering and Economics

In this appendix, we explain the analogy between engineering and economics that underlies our general PPF model. We use the concepts of voltage, current, resistance, and power from engineering and relate them to the concepts of price, trade, scarcity, and utility from economics.

- Voltage: Voltage is the difference in electric potential between two points in a circuit. It is measured in volts (V) and represents the amount of energy per unit charge that is available to move the charge from one point to another. In economics, we can think of voltage as analogous to price, which is the difference in value between two goods or services in a market. Price is measured in units of currency per unit of good or service and represents the amount of utility per unit of good or service that is available to move the good or service from one agent to another.

- Current: Current is the rate of flow of electric charge in a circuit. It is measured in amperes (A) and represents the amount of charge that passes through a point in a circuit per unit time. In economics, we can think of current as analogous to trade, which is the rate of exchange of goods or services in a market. Trade is measured in units of good or service per unit time and represents the amount of good or service that passes through a point in a market per unit time.

- Resistance: Resistance is the opposition to the flow of electric charge in a circuit. It is measured in ohms (Ω) and represents the amount of energy that is dissipated as heat per unit charge that passes through a point in a circuit. In economics, we can think of resistance as analogous to scarcity, which is the limitation to the availability of goods or services in a market. Scarcity is measured in units of currency per unit of good or service and represents the amount of utility that is lost as opportunity cost per unit of good or service that passes through a point in a market.

- Power: Power is the rate of transfer of energy in a circuit. It is measured in watts (W) and represents the amount of energy that is delivered or consumed by a point in a circuit per unit time. In economics, we can think of power as analogous to utility, which is the measure of satisfaction or well-being that an agent derives from consuming goods or services. Utility is measured in units of utility (U) and represents the amount of satisfaction or well-being that is delivered or consumed by an agent per unit time.

Using these analogies, we can interpret our general PPF model as follows:

- The production possibility frontier (PPF) represents the set of all possible combinations of output (y) and input (x) that an economy can produce using its available resources (K and L). - The standard PPF model assumes that there is a constant negative trade-off between output and input, i.e., producing more output requires sacrificing more input. This implies that there is a constant negative slope and a constant concavity for the PPF curve. - The general PPF model relaxes this assumption by introducing a resistance function (R) that depends on the output level. This implies that there is a variable trade-off between output and input, i.e., producing more output may require sacrificing more or less input depending on the resistance level. This implies that there is a variable slope and curvature for the PPF curve. - The efficiency function (η) represents the ratio of power to voltage squared for each point on the PPF curve. This measures how well an economy utilizes its available energy (price) to produce output (utility). - The standard PPF model assumes that there is a constant efficiency level for each point on the PPF curve, i.e., producing more output does not affect the efficiency level. This implies that there is no difference between the standard PPF curve and its efficient subset. - The general PPF model relaxes this assumption by introducing a variable efficiency level for each point on the PPF curve, i.e., producing more output may affect the efficiency level depending on the resistance level. This implies that there is a difference between the general PPF curve and its efficient subset ($P'(Y)$).