Systemic Feedback Equilibria: Co-Evolution of Strategies and Rules in Strategic Settings

Kweku A. Opoku-Agyemang*

March 14, 2025

Abstract

This paper introduces Systemic Feedback Equilibria (SFE), a novel solution concept in game theory where player strategies and the underlying game structure co-evolve. Departing from static frameworks, we model games in which payoffs and action sets adjust endogenously based on aggregate player behavior, capturing the dynamic interplay between individual optimization and systemic adaptation. We formalize SFE as a fixed point where no player unilaterally deviates from their strategy, and the game's rules stabilize under a feedback mapping. Existence is established under continuity and compactness conditions, while uniqueness holds with additional concavity restrictions on the feedback function. We characterize SFE in a class of resource allocation games, demonstrating how rational play can entrench collectively suboptimal outcomes—a phenomenon we term "systemic lock-in." Comparative statics reveal that equilibrium welfare is highly sensitive to the feedback rule's curvature, offering a formal basis to distinguish individual responsibility from structural design. Applied to a tax compliance setting, the model predicts persistent evasion when enforcement adapts sluggishly, aligning with empirical stylized facts. These results suggest SFE provides a unifying lens for phenomena where traditional equilibria fail to account for rule endogeneity, with implications for institutional design and policy analysis.

*Development Economics X. Email: kweku@developmenteconomicsx.com. The author is solely responsible for this article and its implications, and the perspectives therein should not be ascribed to any other person or any organization.

Contents

1 Introduction

The canonical framework of game theory assumes a fixed strategic environment: players optimize within a given set of rules—payoffs, actions, and information—yielding equilibria such as Nash or subgame perfect solutions. Yet, in many economic settings, the rules themselves are not immutable. Market regulations adapt to trader behavior, tax codes evolve with evasion patterns, and social norms shift under collective action. This interdependence between individual strategies and the game's structure challenges standard analyses, as outcomes may reflect not only player choices but also the system's endogenous response. Motivated by the adage "Don't hate the player, hate the game" we propose a new solution concept—Systemic Feedback Equilibria (SFE)—to study games where strategies and rules co-evolve.

In traditional models, the separation of player agency from structural design simplifies analysis but obscures a critical dynamic: rational play can entrench suboptimal systems. Consider a common-pool resource game: if overexploitation triggers stricter quotas, the resulting equilibrium may depend as much on the quota adjustment rule as on player greed. Existing approaches, such as evolutionary game theory (Smith, 1982) or mechanism design (Myerson, 1981), address related ideas but typically treat either strategies or rules as exogenous. Evolutionary models focus on population dynamics under fixed payoffs, while mechanism design optimizes rules for desired outcomes. Neither fully captures settings where players and the game adapt concurrently in a closed loop.

This paper introduces SFE as a fixed point where strategies form a Nash equilibrium given the current rules, and the rules stabilize under a feedback function mapping aggregate behavior to the game's structure. We model this co-evolution explicitly, defining the game as a tuple of strategies and a rule adjustment process. Our main results establish conditions for SFE existence—relying on continuity and compactness of the feedback mapping—and uniqueness, which requires concavity restrictions. We further characterize SFE properties in a parameterized resource allocation game, revealing a phenomenon we call "systemic lock-in": individually rational strategies perpetuate inefficient rules, yielding lower welfare than static benchmarks.

The contribution is threefold. First, SFE extends game theory by integrating rule endogeneity into equilibrium analysis, offering a framework to study systems where structure is neither fixed nor fully controlled. Second, we provide a mathematical foundation—via fixed-point theorems and comparative statics—to dissect how feedback dynamics shape outcomes, formalizing the intuition that blame may lie with the "game" rather than the "players." Third, an application to tax compliance illustrates practical relevance: when enforcement adjusts slowly to evasion, SFE predicts persistent noncompliance, consistent with stylized facts (Andreoni et al., 1998).

Our findings have implications for institutional economics and policy design. By quantifying the role of feedback in equilibrium selection, SFE highlights when structural reform, rather than incentivizing players, is welfare-enhancing. This resonates with debates on market regulation, environmental policy, and organizational behavior, where systemic flaws often overshadow individual intent.

The SFE framework departs fundamentally from standard dynamic game theory, which analyzes strategic interactions where players act sequentially or repeatedly, conditioning decisions on past actions and anticipated responses. In dynamic games—such as repeated games (Fudenberg and Tirole, 1991), stochastic games (Shapley, 1953), or extensive-form games with perfect information (Selten, 1975)—the structure (payoffs, states, transition rules) remains exogenous, with dynamics driven by intertemporal strategy adjustments or equilibrium refinements like subgame perfection (Kreps and Wilson, 1982). SFE, by contrast, endogenizes the game's rules themselves, allowing aggregate behavior to reshape the strategic environment within a single equilibrium concept. This co-evolutionary approach captures settings where the system adapts—e.g., regulations tightening as exploitation rises—beyond the foresight, repetition, or Markovian state transitions of dynamic models (Maskin and Tirole, 2001).

This distinction positions SFE against a rich literature while underscoring its originality. Evolutionary game theory (Smith and Price, 1973; Weibull, 1995) explores strategy adaptation under fixed payoffs via replicator dynamics or mutation-selection processes (Kandori et al., 1993), but the game remains static. Adaptive learning models (Milgrom and Roberts, 1991; Young, 1993) allow players to adjust beliefs or strategies over time, yet the environment is exogenous, unlike SFE's rule evolution. Mechanism design optimizes rules ex ante—e.g., Vickrey's (1961) auctions, Maskin's (1999) implementation theory, or Aghion et al.'s (2010) dynamic mechanisms—assuming a designer's control over the game's structure, while endogenous institution models (Aoki, 2001; Greif and Laitin, 2004) treat rule shifts as emergent from coordination or historical paths, not direct strategic feedback. Recent work on coupled learning (Arieli and Young, 2016) or co-adaptive games (Hart and Mas-Colell, 2003) comes closer, modeling joint strategy-rule adjustment, but typically through decentralized imitation or bounded rationality, not SFE's closed-loop stability. By formalizing a system where rules and strategies co-stabilize, SFE offers a novel lens for phenomena—like persistent inefficiency or regulatory inertia—that static, sequentially dynamic, or learning-based frameworks (Fudenberg and Levine, 1998) struggle to explain.

The paper proceeds as follows. Section 2 defines the SFE framework and proves existence and uniqueness. Section 3 analyzes a resource allocation game, deriving systemic lock-in and comparative statics. Section 4 applies the model to tax compliance, comparing SFE predictions to empirical patterns. Section 5 discusses extensions, including multi-stage feedback and stochastic rules. Section 6 concludes. Proofs are relegated to the Appendix.

2 Model

This section formalizes Systemic Feedback Equilibria (SFE), a solution concept where player strategies and game rules co-evolve. We define the framework, establish existence and uniqueness conditions, and discuss properties distinguishing SFE from standard equilibria.

2.1 Model Setup

Consider a game with a finite set of players $N = \{1, \ldots, n\}$. Each player $i \in N$ chooses a strategy s_i from a compact, convex set $S_i \subset \mathbb{R}^k$, with $S = \prod_{i \in N} S_i$ the joint strategy space. The game's structure is parameterized by a rule vector $r \in R$, where $R \subset \mathbb{R}^m$ is compact and convex, representing payoffs, constraints, or other features (e.g., tax rates, resource quotas). Player i's payoff is $u_i(s, r) : S \times R \to \mathbb{R}$, assumed continuous in $s = (s_1, \ldots, s_n)$ and r.

Unlike static games, r is not fixed exogenously. Instead, rules adjust via a feedback function $f : S \to R$, continuous in s, mapping aggregate strategies to the game's structure. For example, in a resource game, $f(s)$ might increase penalties as total extraction $\sum_i s_i$ rises. The game is thus a tuple (N, S, R, u, f), where $u = (u_1, \ldots, u_n)$.

2.2 Definition of SFE

A Systemic Feedback Equilibrium is a pair $(s^*, r^*) \in S \times R$ satisfying two conditions:

1. Strategic Stability: Given r^*, s^* is a Nash equilibrium:

$$u_i(s_i^*, s_{-i}^*, r^*) \geq u_i(s_i, s_{-i}^*, r^*) \quad \forall s_i \in S_i, \forall i \in N,$$

where $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$.

2. Systemic Consistency: The rules stabilize under feedback:

$$r^* = f(s^*).$$

Intuitively, players optimize within the current game, and the game's rules align with their collective behavior. Define the correspondence $\Gamma : R \rightrightarrows S$ by $\Gamma(r) = \{s \in S \mid s \text{ is a Nash equilibrium given } r\}$, assumed nonempty (e.g., via continuity of u and compactness of S). Then, SFE is a fixed point of the composite mapping $F = f \circ \Gamma : R \rightrightarrows R$, where (s^*, r^*) satisfies $s^* \in \Gamma(r^*)$ and $r^* = f(s^*)$.

2.3 Existence

Theorem 2.1: An SFE exists if u_i is continuous in (s, r) and f is continuous in s.

Proof: Since S_i and R are compact and convex, and u_i is continuous, $\Gamma(r)$ is nonempty, compact-valued, and upper hemicontinuous by the maximum theorem (Berge, 1963). The mapping $F = f \circ \Gamma$ is thus upper hemicontinuous and compact-valued. As R is compact and convex, Kakutani's fixed-point theorem applies, ensuring $r^* \in F(r^*)$. For such r^*, select $s^* \in \Gamma(r^*)$ with $r^* = f(s^*)$, yielding an SFE. Q.E.D.

2.4 Uniqueness

Theorem 2.2: If u_i is concave in s_i, f is Lipschitz continuous with constant $L < 1$, and Γ is single-valued, then SFE is unique.

Proof: Concavity of u_i and compactness of S_i ensure $\Gamma(r)$ is single-valued and continuous (Mas-Colell et al., 1995). Define $H(r) = f(\Gamma(r))$. If $\|f(s) - f(s')\| \leq L\|s - s'\|$ with $L < 1$, and Γ is continuous, H is a contraction on R (by the Banach fixed-point theorem). Thus, H has a unique fixed point r^*, and $s^* = \Gamma(r^*)$ is unique, implying a unique (s^*, r^*). Q.E.D.

2.5 Discussion

SFE differs from Nash equilibria by coupling strategies and rules, introducing a feedback loop absent in static or sequential frameworks. Where Nash assumes a fixed r, SFE solves for r endogenously, potentially amplifying inefficiency if f reinforces suboptimal play (Section 3). Stability hinges on f's responsiveness: if f is Lipschitz with $L < 1$, a tâtonnement process $r_{t+1} = f(s_t(r_t))$ converges to r^* (Fudenberg and Levine, 1998), unlike dynamic games relying on repetition. Efficiency depends on f's alignment with social optima—misaligned feedback may yield "systemic lock-in." The framework's robustness to discontinuous f remains an open question, though upper hemicontinuity suffices for existence (Glicksberg, 1952).

3 Analysis of a Resource Allocation Game

This section applies the Systemic Feedback Equilibrium (SFE) framework to a resource allocation game, illustrating its mechanics and economic implications. We characterize SFE, identify "systemic lock-in," and derive comparative statics to highlight the role of feedback dynamics.

3.1 Setup

Consider n symmetric players exploiting a common resource (e.g., fishery, bandwidth). Player i chooses extraction $s_i \in S_i = [0, \bar{s}]$, where $\bar{s} > 0$ is the capacity constraint, and $s = (s_1, \ldots, s_n) \in S = [0, \bar{s}]^n$. The rule parameter $r \in R = [0, \bar{r}]$ represents a penalty rate (e.g., fines, taxes), with $\bar{r} > 0$. Payoffs are:

$$u_i(s, r) = s_i - \frac{1}{2n} \left(\sum_{j=1}^{n} s_j \right)^2 - r s_i,$$

where s_i is the direct benefit, $\frac{1}{2n}\left(\sum_j s_j\right)^2$ is a quadratic congestion cost shared equally, and rs_i is the penalty. The feedback function adjusts r based on total extraction:

$$r = f(s) = \alpha\left(\sum_{j=1}^{n} s_j\right),$$

with $\alpha > 0$ a sensitivity parameter. If total extraction $\sum_j s_j$ exceeds $\bar{r}/\alpha$, set $f(s) = \bar{r}$ (capping r at its maximum). This captures a regulator increasing penalties as overuse rises, common in resource management.

3.2 SFE Characterization

In SFE, (s^*, r^*) satisfies strategic stability and systemic consistency. Given r, player i maximizes $u_i(s_i, s_{-i}, r)$. The first-order condition (interior solution, as S_i is compact) is:

$$\frac{\partial u_i}{\partial s_i} = 1 - \frac{1}{n}\sum_{j=1}^{n} s_j - r = 0.$$

Symmetry implies $s_i = s$ for all i, so $\sum_j s_j = ns$, and:

$$1 - s - r = 0 \quad \Rightarrow \quad s = 1 - r.$$

Systemic consistency requires $r^* = f(s^*) = \alpha n s^*$. Substituting $s^* = 1 - r^*$:

$$r^* = \alpha n(1 - r^*).$$

Solving:

$$r^* + \alpha n r^* = \alpha n \quad \Rightarrow \quad r^*(1 + \alpha n) = \alpha n \quad \Rightarrow \quad r^* = \frac{\alpha n}{1 + \alpha n}.$$

Then:

$$s^* = 1 - r^* = 1 - \frac{\alpha n}{1 + \alpha n} = \frac{1}{1 + \alpha n}.$$

Check bounds: $s^* > 0$ since $\alpha, n > 0$, and $s^* \leq \bar{s}$ if $\bar{s} \geq 1$; $r^* \leq \bar{r}$ if $\bar{r} \geq \alpha n/(1 + \alpha n)$. Assume these hold (relaxed in Appendix A if needed). Thus, the SFE is:

$$(s^*, r^*) = \left(\frac{1}{1 + \alpha n}, \frac{\alpha n}{1 + \alpha n} \right).$$

3.3 Systemic Lock-In

Compare SFE to the social optimum, maximizing total welfare $W = \sum_i u_i = ns - \frac{1}{2}ns^2 - rns$. With $r = \alpha ns$:

$$W(s) = ns - \frac{1}{2}ns^2 - \alpha n^2 s^2.$$

The optimum s^o satisfies:

$$\frac{dW}{ds} = n - ns - 2\alpha n^2 s = 0 \quad \Rightarrow \quad s^o = \frac{1}{1 + 2\alpha n}.$$

Since $s^* = 1/(1+\alpha n) < s^o$ for $\alpha n > 0$, SFE under-extracts relative to the optimum. Total extraction $ns^* = n/(1+\alpha n)$ falls as α or n rises, yet congestion ns^2 and penalties rns persist, reducing welfare below $W(s^o)$. We term this "systemic lock-in": rational play, amplified by feedback, entrenches inefficiency. For large n or α, $s^* \to 0$, collapsing resource use—a tragedy of over-regulation.

3.4 Comparative Statics

How does SFE respond to α? Compute:

$$\frac{\partial s^*}{\partial \alpha} = -\frac{n}{(1 + \alpha n)^2} < 0, \quad \frac{\partial r^*}{\partial \alpha} = \frac{n}{(1 + \alpha n)^2} > 0.$$

Higher α (stronger feedback) reduces extraction but raises penalties, tightening the system. Welfare $W(s^*) = ns^* - \frac{1}{2}n(s^*)^2 - \alpha n^2(s^*)^2$ decreases in α (verified numerically), as over-penalization dominates. Contrast with a static Nash equilibrium at fixed $r = 0$: $s = 1$, over-extracting but avoiding feedback-induced collapse.

The resource game reveals SFE's power: feedback can invert the tragedy of the commons into underuse, driven not by player intent but by systemic overreaction. This aligns with fishery quotas or bandwidth throttling, where rules tighten excessively. Section 4 explores a tax evasion analog.

4 Application to Tax Compliance

This section applies Systemic Feedback Equilibria (SFE) to a tax compliance game, modeling how evasion and enforcement co-evolve. We derive SFE, analyze its implications, and connect to stylized empirical facts, illustrating the framework's relevance to policy design.

4.1 Setup

Consider n symmetric taxpayers, each with income normalized to 1. Player i chooses evasion $s_i \in S_i = [0,1]$, where s_i is the fraction of income concealed, and $s = (s_1, \ldots, s_n) \in S = [0,1]^n$. The rule parameter $r \in R = [0, \bar{r}]$ is the audit probability, with $\bar{r} \leq 1$. Payoffs are:

$$u_i(s, r) = 1 - t(1 - s_i) - r\phi s_i,$$

where $t \in (0,1)$ is the tax rate, $1 - t(1 - s_i)$ is after-tax income (evading s_i saves ts_i), and $r\phi s_i$ is the expected penalty, with $\phi > t$ the fine rate (e.g., recovered tax plus interest). The feedback function adjusts audits based on aggregate evasion:

$$r = f(s) = \alpha \left(\sum_{j=1}^{n} s_j \right),$$

capped at $\bar{r}$ if $\alpha \sum_j s_j > \bar{r}$, where $\alpha > 0$ reflects enforcement responsiveness (e.g., IRS audits rising with detected evasion).

4.2 SFE Characterization

In SFE, (s^*, r^*) satisfies strategic stability and systemic consistency. Given r, player i maximizes $u_i(s_i, s_{-i}, r)$. The first-order condition (interior if $t < r\phi$) is:

$$\frac{\partial u_i}{\partial s_i} = t - r\phi = 0 \quad \Rightarrow \quad r = \frac{t}{\phi}.$$

However, r is endogenous. Symmetry gives $s_i = s$, so $\sum_j s_j = ns$, and $r = f(s) = \alpha ns$. If $t - r\phi > 0$ (low audits), $s_i = 1$ (corner solution); if $t - r\phi < 0$, $s_i = 0$. Assume an interior SFE: $t = r\phi$. Then:

$$r^* = \alpha ns^* \quad \text{and} \quad t = r^*\phi.$$

Substitute:

$$t = (\alpha ns^*)\phi \quad \Rightarrow \quad s^* = \frac{t}{\alpha n\phi}.$$

Thus:

$$r^* = \alpha ns^* = \frac{t}{\phi}.$$

Verify bounds: $s^* \leq 1$ if $\alpha n\phi \geq t$, and $r^* \leq \bar{r}$ if $\bar{r} \geq t/\phi$. Assume these hold (relaxed in Appendix B if needed). The SFE is:

$$(s^*, r^*) = \left(\frac{t}{\alpha n\phi}, \frac{t}{\phi}\right).$$

4.3 Implications and Systemic Lock-In

Aggregate evasion is $ns^* = t/(\alpha\phi)$. For $\alpha\phi < t$, $s^* > 1$, implying a corner solution ($s^* = 1, r^* = \alpha n$), but we focus on interior cases. Welfare (tax revenue minus enforcement costs, say cr, with $c > 0$) is:

$$W = tn(1 - s^*) - cr^* = tn\left(1 - \frac{t}{\alpha n\phi}\right) - c\frac{t}{\phi}.$$

Compare to a static Nash at fixed $r = t/\phi$: $s = 0$ (full compliance), yielding $W = tn$. In SFE, $s^* > 0$, reducing revenue due to feedback-driven audits. This "systemic lock-in" reflects a self-reinforcing cycle: evasion triggers audits, which deter evasion but sustain $r^* > 0$, unlike the no-evasion static ideal. Stronger feedback (high α) lowers s^* but raises r^*, balancing deterrence and cost.

4.4 Empirical Alignment

SFE predicts persistent evasion ($ns^* > 0$) despite enforcement, matching stylized facts: U.S. tax gaps hover at 15% of liability (IRS, 2021), suggesting $\alpha\phi$ fails to eliminate s^*. Slow audit adjustments (α small) yield higher s^*, consistent with lagged IRS responses (Andreoni et al., 1998). Comparative statics show:

$$\frac{\partial s^*}{\partial \alpha} = -\frac{t}{n\phi\alpha^2} < 0, \quad \frac{\partial r^*}{\partial \alpha} = 0,$$

implying faster enforcement reduces evasion without altering r^*, aligning with deterrence studies (Slemrod, 2007).

The tax game underscores SFE's insight: persistent noncompliance may stem from the "game" (feedback dynamics) rather than "players" (taxpayers). Policy could target α or ϕ, not just individual behavior, echoing Section 1's motivation.

5 Extensions

This section explores three extensions of the Systemic Feedback Equilibrium (SFE) framework: multi-stage feedback, stochastic rules, and asymmetric players. These broaden the model's applicability to dynamic, uncertain, and heterogeneous settings, preserving its co-evolutionary core.

5.1 Multi-Stage Feedback

In the baseline SFE, rules adjust instantaneously via $r = f(s)$. Consider a multi-stage setting where feedback unfolds over discrete periods. At stage $t = 0, 1, \ldots, T$, players choose $s^t = (s_1^t, \ldots, s_n^t) \in S$, and rules evolve as:

$$r^{t+1} = f(s^t, r^t),$$

with $r^0 \in R$ given. Payoffs are $u_i(s^t, r^t)$, and players maximize discounted utility $\sum_{t=0}^{T} \delta^t u_i(s^t, r^t)$, where $\delta \in (0, 1)$. An SFE is a sequence $\{(s^{t*}, r^{t*})\}_{t=0}^{T}$ where, for each t:

- s^{t*} is a Nash equilibrium given r^{t*} and continuation values,
- $r^{t+1*} = f(s^{t*}, r^{t*})$, with $r^{0*} = r^0$.

For the Section 3 resource game, let $f(s^t, r^t) = (1 - \beta)r^t + \beta\alpha \sum_j s_j^t$, where $\beta \in (0, 1)$ governs adjustment speed. In a stationary SFE ($T \to \infty$), $s^{t*} = s^*$, $r^{t*} = r^*$, converging to $(1/(1 + \alpha n), \alpha n/(1 + \alpha n))$ if $\beta = 1$. Slower adjustment ($\beta < 1$) mitigates lock-in, potentially raising welfare. Existence follows from continuity and compactness (Appendix C).

5.2 Stochastic Rules

Now suppose feedback is noisy. Let $r = f(s, \epsilon)$, where $\epsilon \sim F$ on $[-\bar{\epsilon}, \bar{\epsilon}]$, and $f(s, \epsilon) = \alpha \sum_j s_j + \epsilon$, capped at $R = [0, \bar{r}]$. Players maximize expected utility:

$$\mathbb{E}_\epsilon[u_i(s, f(s, \epsilon))].$$

An SFE is (s^*, r^*) where s^* is a Nash equilibrium under expected payoffs, and $r^* = \mathbb{E}[f(s^*, \epsilon)]$. For the tax game (Section 4), $u_i = 1 - t(1 - s_i) - f(s, \epsilon)\phi s_i$. Symmetry gives $s_i = s$, and:

$$t - \phi(\alpha n s + \mathbb{E}[\epsilon]) = 0 \quad \Rightarrow \quad s^* = \frac{t - \phi\mathbb{E}[\epsilon]}{\alpha n \phi}.$$

If $\mathbb{E}[\epsilon] = 0$, $s^* = t/(\alpha n \phi)$. Variance σ_ϵ^2 affects risk-averse players (Appendix D), reducing s^*. Existence holds via continuity (Appendix C).

5.3 Asymmetric Players

Relax symmetry, allowing heterogeneity in player types. In the resource game, suppose player i has cost parameter $c_i > 0$, with payoffs:

$$u_i(s, r) = s_i - c_i \left(\sum_{j=1}^{n} s_j \right)^2 - r s_i,$$

where c_i reflects sensitivity to congestion (e.g., small vs. large firms). Feedback remains $r = f(s) = \alpha \sum_j s_j$. In SFE, s_i^* satisfies:

$$1 - 2c_i \sum_j s_j^* - r^* = 0 \quad \Rightarrow \quad s_i^* = \frac{1 - r^*}{2c_i \sum_j s_j^*}.$$

Define $S^* = \sum_j s_j^*$. Then $r^* = \alpha S^*$, and:

$$S^* = \sum_i \frac{1 - r^*}{2c_i S^*} = \frac{1 - r^*}{2S^*} \sum_i \frac{1}{c_i}.$$

Let $\gamma = \sum_i 1/c_i$. Systemic consistency gives:

$$r^* = \alpha \frac{1 - r^*}{2r^*/\alpha} \gamma = \frac{\alpha\gamma(1 - r^*)}{2r^*}.$$

Solving $r^* = \alpha\gamma(1 - r^*)/(2r^*)$:

$$2(r^*)^2 = \alpha\gamma - \alpha\gamma r^* \quad \Rightarrow \quad 2(r^*)^2 + \alpha\gamma r^* - \alpha\gamma = 0.$$

The positive root is:

$$r^* = \frac{-\alpha\gamma + \sqrt{(\alpha\gamma)^2 + 8\alpha\gamma}}{4}, \quad S^* = \frac{r^*}{\alpha}.$$

Then $s_i^* = (1 - r^*)/(2c_i S^*)$. High-$c_i$ players extract less, yet feedback ties r^* to total use, amplifying lock-in if γ (heterogeneity) grows. Existence is assured (Appendix C).

5.4 Discussion

Multi-stage SFE captures regulatory lag, stochastic SFE models uncertainty, and asymmetric SFE reflects heterogeneous agents—all retaining the "game-driven" insight. Future work could explore non-stationary paths, correlated shocks, or learning dynamics (Appendix E).

6 Conclusion

This paper introduces Systemic Feedback Equilibria (SFE), a game-theoretic framework where strategies and rules co-evolve, inspired by the adage "Don't hate the player, hate the game." Unlike static or dynamic models with exogenous structures, SFE captures settings where aggregate behavior shapes the game itself—be it resource penalties, tax audits, or institutional norms. We establish existence and uniqueness under standard conditions, demonstrate "systemic lock-in" in a resource allocation game, and align tax compliance predictions with empirical patterns. Extensions to multi-stage, stochastic, and asymmetric settings broaden the framework's scope, revealing how feedback dynamics amplify inefficiency beyond individual intent.

SFE offers a dual contribution: theoretically, it extends equilibrium analysis to endogenous rule formation, bridging gaps in evolutionary and mechanism design literatures; practically, it reframes policy debates—suggesting structural reform over agent-targeted incentives in systems like taxation or resource management. The framework's flexibility invites applications to markets, environmental policy, or organizational design, where feedback loops are pervasive yet understudied.

By formalizing the interplay of play and structure, SFE underscores a core insight: when outcomes falter, the game, not the players, may bear the blame.

7 References

Aghion, P., A. Alesina, and F. Trebbi (2010). "Endogenous Political Institutions." Quarterly Journal of Economics, 125(2), 565–612.

Andreoni, J., B. Erard, and J. Feinstein (1998). "Tax Compliance." Journal of Economic Literature, 36(2), 818–860.

Aoki, M. (2001). Toward a Comparative Institutional Analysis. MIT Press.

Arieli, I., and H. P. Young (2016). "Stochastic Learning Dynamics and Speed of Convergence in Population Games." Econometrica, 84(2), 627–676.

Berge, C. (1963). Topological Spaces. Oliver and Boyd.

Fan, K., and I. Glicksberg (1952). "Fixed-Point and Minimax Theorems in Locally Convex Topological Linear Spaces." Proceedings of the National Academy of Sciences, 38(2), 121–126.

Fudenberg, D., and D. K. Levine (1998). The Theory of Learning in Games. MIT Press.

Fudenberg, D., and J. Tirole (1991). Game Theory. MIT Press.

Greif, A., and D. D. Laitin (2004). "A Theory of Endogenous Institutional Change." American Political Science Review, 98(4), 633–652.

Hart, S., and A. Mas-Colell (2003). "Regret-Based Continuous-Time Dynamics." Games and Economic Behavior, 45(2), 375–394.

Hurwicz, L. (1973). "The Design of Mechanisms for Resource Allocation." American Economic Review, 63(2), 1–30.

Internal Revenue Service (2021). "Federal Tax Compliance Research: Tax Gap Estimates for Tax Years 2011–2013." Publication 1415.

Kandori, M., G. J. Mailath, and R. Rob (1993). "Learning, Mutation, and Long Run Equilibria in Games." Econometrica, 61(1), 29–56.

Kreps, D. M., and R. Wilson (1982). "Sequential Equilibria." Econometrica, 50(4), 863–894.

Maskin, E. (1999). "Nash Equilibrium and Welfare Optimality." Review of Economic Studies, 66(1), 23–38.

Maskin, E., and J. Tirole (2001). "Markov Perfect Equilibrium: I. Observable Actions." Journal of Economic Theory, 100(2), 191–219.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995). Microeconomic Theory. Oxford University Press.

Milgrom, P., and J. Roberts (1991). "Adaptive and Sophisticated Learning in Normal Form Games." Games and Economic Behavior, 3(1), 82–100.

Myerson, R. B. (1981). "Optimal Auction Design." Mathematics of Operations Research, 6(1), 58–73.

Selten, R. (1975). "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games." International Journal of Game Theory, 4(1), 25–55.

Shapley, L. S. (1953). "Stochastic Games." Proceedings of the National Academy of Sciences, 39(10), 1095–1100.

Slemrod, J. (2007). "Cheating Ourselves: The Economics of Tax Evasion." Journal of Economic Perspectives, 21(1), 25–48.

Smith, J. M. (1982). Evolution and the Theory of Games. Cambridge University Press.

Smith, J. M., and G. R. Price (1973). "The Logic of Animal Conflict." Nature, 246(5427), 15–18.

Vickrey, W. (1961). "Counterspeculation, Auctions, and Competitive Sealed Tenders." Journal of Finance, 16(1), 8–37.

Weibull, J. W. (1995). Evolutionary Game Theory. MIT Press.

Young, H. P. (1993). "The Evolution of Conventions." Econometrica, 61(1), 57–84.

Appendices

Appendix Overview

This appendix supplements the main text with technical details and extensions for the Systemic Feedback Equilibrium (SFE) framework. It comprises five parts:

- Appendix A: Relaxes bounds ($\bar{s}$, $\bar{r}$) and allows non-convex strategy sets (S_i) in the Section 3 resource game, proving existence using Fan-Glicksberg (1952).
- Appendix B: Addresses corner cases and bound relaxations ($\alpha n\phi \geq t$) for the Section 4 tax game, with generalized existence.
- Appendix C: Provides existence proofs for Section 5's multi-stage and stochastic extensions under continuity and compactness.
- Appendix D: Incorporates risk aversion into the stochastic SFE (Section 5.2), adjusting the tax game solution.
- Appendix E: Sketches a learning extension where players adapt beliefs about the feedback function, building on Section 5.

Proofs and derivations are detailed to ensure rigor, supporting the main text's claims.

Appendix A: Resource Game with Relaxed Assumptions

Section 3 assumes $s^* \leq \bar{s}$ and $r^* \leq \bar{r}$. Relax these: let $S_i = [0, \infty)$ (unbounded) or non-convex (e.g., $S_i = [0, 1] \cup [2, 3]$), and $R = [0, \infty)$. Payoffs remain $u_i = s_i - \frac{1}{2n}(\sum_j s_j)^2 - rs_i$, with $f(s) = \alpha \sum_j s_j$.

For unbounded S_i, assume s_i has an implicit upper bound via payoffs (e.g., $u_i < 0$ if s_i large). The FOC $1-s-r = 0$ holds interiorly, and $r^* = \alpha n s^*$ gives $s^* = 1/(1+\alpha n)$, as before, if feasible. For non-convex S_i, $\Gamma(r)$ may be multi-valued, but $f \circ \Gamma$ remains upper hemicontinuous. Fan-Glicksberg (1952) ensures a fixed point $r^* \in f(\Gamma(r^*))$, with $s^* \in \Gamma(r^*)$, if u_i is continuous and S compact in the product topology.

Appendix B: Tax Game with Corner Cases

Section 4 assumes $\alpha n\phi \geq t$ for interior $s^* \leq 1$. If $\alpha n\phi < t$, $s^* = t/(\alpha n\phi) > 1$, hitting the corner $s^* = 1$, $r^* = \alpha n$. Payoff $u_i = 1 - t(1 - s_i) - r\phi s_i$ gives:

- If $t - r\phi > 0$, $s_i = 1$,
- If $t - r\phi < 0$, $s_i = 0$.

At $s^* = 1$, $r^* = \alpha n < t/\phi$ (if $\alpha n\phi < t$), consistent with evasion persisting. Existence holds: $\Gamma(r)$ is compact-valued, and f continuous, per Theorem 2.1. Unbounded R requires r capped implicitly by f's range.

Appendix C: Existence for Extensions

For multi-stage SFE (Section 5.1), $s^{t*} \in \Gamma(r^{t*})$, $r^{t+1*} = f(s^{t*}, r^{t*})$ forms a Markov process. Continuity of u_i and f, compactness of S and R, and finite T ensure a solution via backward induction. For $T \to \infty$, stationary SFE exists under stationarity of f (Maskin and Tirole, 2001).

For stochastic SFE (Section 5.2), $\mathbb{E}[u_i]$ is continuous in s, and S compact, yielding a Nash equilibrium s^*. Then $r^* = \mathbb{E}[f(s^*, \epsilon)]$ is well-defined if f is bounded, satisfying Theorem 2.1's logic.

Appendix D: Risk Aversion in Stochastic SFE

Add risk aversion to Section 5.2's tax game. Let utility be concave, e.g., $v_i(z) = -\exp(-\theta z)$, $\theta > 0$, over $z = 1 - t(1 - s_i) - f(s, \epsilon)\phi s_i$. Maximize:

$$\mathbb{E}[v_i] = -\mathbb{E}[\exp(-\theta(1 - t + ts_i - (\alpha ns + \epsilon)\phi s_i))].$$

FOC approximates (via certainty equivalent):

$$t - \phi(\alpha ns^* + \mathbb{E}[\epsilon]) + \theta\phi^2\sigma_\epsilon^2 s^* = 0.$$

If $\mathbb{E}[\epsilon] = 0$, $s^* = t/(\alpha n\phi + \theta\phi^2\sigma_\epsilon^2) < t/(\alpha n\phi)$, reducing evasion as risk aversion (θ) or variance (σ_ϵ^2) rises.

Appendix E: Learning Extension

Players may learn f over time. In Section 5's resource game, suppose $f(s) = \alpha \sum_j s_j$, but players perceive $f(s) = \hat{\alpha} \sum_j s_j$, updating $\hat{\alpha}$ via Bayesian inference from observed r^t. Initial SFE uses $\hat{\alpha}$, converging to true s^* as beliefs align (Fudenberg and Levine, 1998). Details require simulation, left for future work.

Appendix A: Resource Game with Relaxed Assumptions

Section 3 assumes $S_i = [0, \bar{s}]$ and $R = [0, \bar{r}]$ are compact and convex, with $s^* \leq \bar{s}$ and $r^* \leq \bar{r}$ ensured by parameter restrictions. Here, we relax these bounds and convexity, proving SFE existence under weaker conditions. The resource game retains payoffs $u_i(s, r) = s_i - \frac{1}{2n}(\sum_{j=1}^n s_j)^2 - rs_i$ and feedback $f(s) = \alpha \sum_{j=1}^n s_j$, with $\alpha > 0$.

A.1 Unbounded Strategy and Rule Sets

First, let $S_i = [0, \infty)$ and $R = [0, \infty)$, removing finite bounds. The payoff u_i decreases for large s_i due to the quadratic congestion term: if $\sum_j s_j$ grows, $u_i \to -\infty$ unless offset by small s_i or r. Define $\Gamma(r) = \{s \in S \mid s \text{ is a Nash equilibrium given } r\}$. The FOC for an interior solution is:

$$\frac{\partial u_i}{\partial s_i} = 1 - \frac{1}{n} \sum_{j=1}^n s_j - r = 0, \tag{1}$$

yielding $s_i = 1 - r - \frac{1}{n} \sum_{j \neq i} s_j$. Symmetry suggests $s_i = s$, so:

$$1 - s - r = 0 \quad \Rightarrow \quad s = 1 - r, \tag{2}$$

and $r = f(s) = \alpha n s$. Solving:

$$r = \alpha n(1 - r) \quad \Rightarrow \quad r(1 + \alpha n) = \alpha n \quad \Rightarrow \quad r^* = \frac{\alpha n}{1 + \alpha n}, \quad s^* = \frac{1}{1 + \alpha n}. \tag{3}$$

Since $s^*, r^* \geq 0$, no artificial bounds are needed if u_i ensures finite choices (e.g., $s_i < \infty$ as u_i becomes negative). Existence requires compactness, so assume an implicit bound (e.g., $S_i = [0, M]$,

$R = [0, M]$, M large), adjusted post-hoc to contain (s^*, r^*). Theorem 2.1 applies: u_i continuous, f continuous, S, R compact yield an SFE.

A.2 Non-Convex Strategy Sets

Now let S_i be compact but non-convex, e.g., $S_i = [0, 1] \cup [2, 3]$, and $R = [0, \bar{r}]$. Convexity of S_i ensures $\Gamma(r)$ is single-valued under concavity (Section 2), but non-convexity makes $\Gamma(r)$ a correspondence. For fixed r, maximize:

$$u_i(s_i, s_{-i}, r) = s_i - \frac{1}{2n}\left(s_i + \sum_{j \neq i} s_j\right)^2 - r s_i. \tag{4}$$

Since u_i is continuous and S_i compact, $\Gamma(r)$ is nonempty and compact-valued. The quadratic term couples s_i and s_{-i}, but u_i remains quasi-concave in s_i over segments (e.g., [0, 1] or [2, 3]). However, multiple equilibria may arise: for small r, $s_i = 3$ might dominate $s_i = 1$ if congestion is low.

Define $F = f \circ \Gamma : R \rightrightarrows R$. Since f is continuous and linear, and Γ is upper hemicontinuous (maximum theorem, Berge, 1963), F is upper hemicontinuous. Fan-Glicksberg (1952) applies: for R compact (not necessarily convex), a fixed point $r^* \in F(r^*)$ exists, with $s^* \in \Gamma(r^*)$ an SFE. Explicit computation is complex due to non-convexity, but the Section 3 solution holds if $s^* \in [0, 1] \subset S_i$.

A.3 Discussion

Unbounded sets rely on payoff-driven bounds, while non-convexity broadens applicability (e.g., discrete effort levels). Both preserve SFE's core insight, with existence robust to these relaxations.

Appendix B: Tax Game with Corner Cases

Section 4 assumes $\alpha n \phi \geq t$ to ensure an interior SFE with $s^* \leq 1$ and $r^* \leq \bar{r}$. Here, we relax this bound, analyze corner solutions, and confirm existence under weaker conditions. The tax game retains $u_i(s, r) = 1 - t(1 - s_i) - r\phi s_i$, $S_i = [0, 1]$, $R = [0, \bar{r}]$, and $f(s) = \alpha \sum_{j=1}^{n} s_j$, with $\alpha, \phi, t > 0$.

B.1 Corner Solutions

If $\alpha n\phi < t$, the interior $s^* = t/(\alpha n\phi) > 1$, exceeding S_i's bound. Compute $\Gamma(r)$: maximize $u_i = 1 - t + ts_i - r\phi s_i$. The FOC is:

$$\frac{\partial u_i}{\partial s_i} = t - r\phi, \tag{5}$$

$$\text{If } t - r\phi > 0, \text{ then } s_i = 1 \text{ (corner)}, \tag{6}$$

$$\text{If } t - r\phi < 0, \text{ then } s_i = 0 \text{ (corner)}, \tag{7}$$

$$\text{If } t - r\phi = 0, \text{ then } s_i \in [0,1] \text{ (indifferent)}. \tag{8}$$

For $r < t/\phi$, $s_i = 1$; for $r > t/\phi$, $s_i = 0$. Symmetry gives $s_i = s$, so:

$$s = 1, \quad r = f(s) = \alpha n, \tag{9}$$

$$s = 0, \quad r = f(s) = 0. \tag{10}$$

Check consistency:

- If $s^* = 1$, $r^* = \alpha n$. Then $t - r^*\phi = t - \alpha n\phi$. If $t > \alpha n\phi$, $s_i = 1$ holds, an SFE: $(s^*, r^*) = (1, \alpha n)$.
- If $s^* = 0$, $r^* = 0$, but $t - 0\phi = t > 0$, so $s_i = 1$, inconsistent.

Thus, for $\alpha n\phi < t$ and $\bar{r} \geq \alpha n$, $(1, \alpha n)$ is the SFE, with full evasion unless $\alpha n > t/\phi$ (then $r^* > t/\phi$, $s^* = 0$, but $r^* = 0$ contradicts). If $\bar{r} < \alpha n$, $r^* = \bar{r}$, and $s^* = 1$ if $\bar{r} < t/\phi$.

B.2 Relaxed Bounds

Let $R = [0, \infty)$, removing $\bar{r}$. For $s^* = 1$, $r^* = \alpha n$, finite if $\alpha n < t/\phi$ fails. If $\alpha n > t/\phi$, interior $s^* = t/(\alpha n\phi)$ holds, but $r^* = t/\phi$ is unbounded unless ϕ caps enforcement implicitly. Assume $R = [0, M]$, $M > t/\phi$, for compactness. The interior SFE persists if $\alpha n\phi \geq t$.

B.3 Generalized Existence

For any $\alpha n\phi$, $\Gamma(r)$ is a step function: $s = 1$ if $r < t/\phi$, $s = 0$ if $r > t/\phi$, $s \in [0, 1]$ if $r = t/\phi$. $\Gamma(r)$ is upper hemicontinuous (right-continuous at t/ϕ), compact-valued. $f(s) = \alpha ns$ is continuous, so $F = f \circ \Gamma : R \rightrightarrows R$ is upper hemicontinuous. Kakutani applies over compact R, ensuring an SFE (e.g., $(1, \alpha n)$ or $(t/(\alpha n\phi), t/\phi)$ if interior).

If $R = [0, \infty)$, existence requires f's range to be bounded or u_i to penalize large r. Alternatively, Fan-Glicksberg (1952) holds with non-convex S_i (e.g., $\{0, 1\}$), as u_i remains continuous.

B.4 Discussion

Corner $s^* = 1$ aligns with persistent evasion when enforcement lags ($\alpha n\phi < t$), reinforcing Section 4's lock-in. Relaxed bounds broaden applicability, with existence robust.

Appendix C: Existence for Extensions

Section 5 introduces multi-stage and stochastic SFE variants. This corresponding appendix proves existence for both under continuity and compactness, extending Theorems 2.1 and 2.2.

C.1 Multi-Stage Feedback

In Section 5.1, players choose $s^t \in S = \prod_i S_i$ at $t = 0, 1, \dots, T$, with S_i compact, convex, and R compact. Payoffs are $u_i(s^t, r^t)$, continuous, and rules evolve via:

$$r^{t+1} = f(s^t, r^t), \tag{11}$$

with r^0 given. Players maximize $\sum_{t=0}^{T} \delta^t u_i(s^t, r^t)$, $\delta \in (0, 1)$. An SFE is $\{(s^{t*}, r^{t*})\}_{t=0}^{T}$ where $s^{t*} \in \Gamma(r^{t*})$ (Nash given r^{t*} and continuation values), and $r^{t+1*} = f(s^{t*}, r^{t*})$.

For finite T, solve by backward induction. At T, $s^{T*} \in \Gamma(r^{T*})$, nonempty and compact since u_i is continuous and S compact (Berge, 1963). Set $r^{T*} = f(s^{T-1*}, r^{T-1*})$. At $t = T - 1$, maximize $u_i(s^{T-1}, r^{T-1}) + \delta u_i(s^{T*}(r^{T-1}), f(s^{T-1}, r^{T-1}))$. Continuity of f and u_i ensures $\Gamma(r^{T-1})$ exists. Inductively, an SFE exists for all t.

For $T \to \infty$, seek a stationary SFE: $s^{t*} = s^*$, $r^{t*} = r^*$. Then $r^* = f(s^*, r^*)$, and $s^* \in \Gamma(r^*)$. Define $F(r) = f(\Gamma(r), r)$. If f is continuous in (s, r), F is upper hemicontinuous (since Γ is), and Kakutani applies over compact R. For $f(s, r) = (1-\beta)r + \beta\alpha \sum_j s_j$, stationarity holds as in Section 3 when $\beta = 1$.

C.2 Stochastic Rules

In Section 5.2, $r = f(s, \epsilon)$, $\epsilon \sim F$ on $[-\bar{\epsilon}, \bar{\epsilon}]$, $f(s, \epsilon) = \alpha \sum_j s_j + \epsilon$, and players maximize:

$$\mathbb{E}_\epsilon[u_i(s, f(s, \epsilon))]. \tag{12}$$

An SFE is (s^*, r^*) with $s^* \in \Gamma(\mathbb{E}[f(s, \epsilon)])$, $r^* = \mathbb{E}[f(s^*, \epsilon)]$. Since u_i is continuous and f linear in s, $\mathbb{E}[u_i]$ is continuous over compact S. The best-response correspondence $B_i(s_{-i}) = \arg\max_{s_i} \mathbb{E}[u_i(s_i, s_{-i}, f(s, \epsilon))]$ is nonempty, compact-valued, and upper hemicontinuous. Kakutani ensures a fixed point s^*, and $r^* = \alpha \sum_j s_j^*$ (if $\mathbb{E}[\epsilon] = 0$).

For unbounded R, assume f's range is compact (e.g., $[0, \bar{r}]$) or u_i penalizes large r. Existence mirrors stochastic games (Shapley, 1953), with r as an expected state.

C.3 Discussion

Both extensions preserve SFE's logic. Multi-stage existence parallels Markov perfect equilibria (Maskin and Tirole, 2001), but with endogenous rules. Stochastic SFE adapts static arguments to uncertainty, robust to noise in f.

Appendix D: Risk Aversion in Stochastic SFE

Section 5.2 assumes risk-neutral players in the stochastic SFE, where $r = f(s, \epsilon) = \alpha \sum_j s_j + \epsilon$. Here, we introduce risk aversion, modifying the tax game from Section 4 to explore its impact on evasion s^*.

D.1 Setup with Risk Aversion

Retain $u_i(s,r) = 1 - t(1-s_i) - r\phi s_i$, $S_i = [0,1]$, and $f(s,\epsilon) = \alpha \sum_j s_j + \epsilon$, with $\epsilon \sim F$ on $[-\bar{\epsilon}, \bar{\epsilon}]$, $\mathbb{E}[\epsilon] = 0$, $\mathrm{Var}(\epsilon) = \sigma_\epsilon^2$. Instead of maximizing $\mathbb{E}[u_i]$, players maximize a concave utility over income $z_i = 1 - t(1-s_i) - f(s,\epsilon)\phi s_i$:

$$V_i(s) = \mathbb{E}[v(z_i)], \tag{13}$$

where $v(z) = -\exp(-\theta z)$, $\theta > 0$ is the risk aversion coefficient (exponential utility ensures constant absolute risk aversion). The SFE is (s^*, r^*) with s^* a Nash equilibrium under V_i, and $r^* = \mathbb{E}[f(s^*, \epsilon)]$.

D.2 SFE Derivation

Symmetry implies $s_i = s$, so $f(s,\epsilon) = \alpha ns + \epsilon$, and:

$$z_i = 1 - t(1-s_i) - (\alpha ns + \epsilon)\phi s_i. \tag{14}$$

Then:

$$V_i(s_i, s_{-i}) = -\mathbb{E}[\exp(-\theta(1 - t + ts_i - (\alpha ns + \epsilon)\phi s_i))]. \tag{15}$$

For $s_{-i} = s$, maximize $V_i(s_i, s)$. The FOC is:

$$\frac{\partial V_i}{\partial s_i} = \theta \mathbb{E}[\exp(-\theta z_i)(t - \phi(\alpha ns + \epsilon))] = 0. \tag{16}$$

Factor out the expectation:

$$\mathbb{E}[\exp(-\theta z_i)(t - \phi(\alpha ns + \epsilon))] = t\mathbb{E}[\exp(-\theta z_i)] - \phi \mathbb{E}[\exp(-\theta z_i)(\alpha ns + \epsilon)] = 0. \tag{17}$$

Since ϵ is independent, approximate for small σ_ϵ^2 (via Taylor expansion around $\epsilon = 0$):

$$\mathbb{E}[\exp(-\theta z_i)] \approx \exp(-\theta(1 - t + ts_i - \alpha ns\phi s_i)) \left(1 + \frac{\theta^2 \phi^2 s_i^2 \sigma_\epsilon^2}{2}\right), \tag{18}$$

$$\mathbb{E}[\exp(-\theta z_i)\epsilon] \approx 0. \tag{19}$$

Thus:

$$t - \phi\alpha ns - \phi\frac{\mathbb{E}[\exp(-\theta z_i)\epsilon]}{\mathbb{E}[\exp(-\theta z_i)]} \approx t - \phi\alpha ns = 0, \tag{20}$$

but risk aversion adjusts this. Using certainty equivalent, $V_i \approx v(\mathbb{E}[z_i] - \frac{\theta}{2}\text{Var}(z_i))$, where:

$$\mathbb{E}[z_i] = 1 - t + ts_i - \alpha ns\phi s_i, \quad \text{Var}(z_i) = \phi^2 s_i^2 \sigma_\epsilon^2. \tag{21}$$

Maximize $\tilde{z}_i = 1 - t + ts_i - \alpha ns\phi s_i - \frac{\theta}{2}\phi^2 s_i^2 \sigma_\epsilon^2$. FOC:

$$t - \phi\alpha ns^* - \theta\phi^2 s^* \sigma_\epsilon^2 = 0, \tag{22}$$

$$s^* = \frac{t}{\alpha n\phi + \theta\phi^2\sigma_\epsilon^2}. \tag{23}$$

Then $r^* = \alpha ns^* = \frac{\alpha nt}{\alpha n\phi + \theta\phi^2\sigma_\epsilon^2}$.

D.3 Implications

Compared to $s^* = \frac{t}{\alpha n\phi}$ (Section 5.2), risk aversion ($\theta > 0$) and variance ($\sigma_\epsilon^2 > 0$) reduce evasion, as the penalty's uncertainty deters risk-averse players. If $\theta = 0$ or $\sigma_\epsilon^2 = 0$, it reverts to the risk-neutral case. Bounds hold if $\alpha n\phi + \theta\phi^2\sigma_\epsilon^2 \geq t$.

D.4 Existence

V_i is continuous (exponential utility is bounded), and S compact, so $\Gamma(\mathbb{E}[f])$ is nonempty and upper hemicontinuous. Kakutani ensures an SFE (Appendix C).

D.5 Discussion

Risk aversion tempers systemic lock-in, aligning with deterrence effects in tax compliance (Slemrod, 2007).

Appendix E: Learning Extension

Section 5 assumes players know the feedback function $f(s)$. Here, we extend SFE to a learning setting where players update beliefs about f based on observed rules, using the Section 3 resource game as a base.

E.1 Setup with Learning

Retain $u_i(s,r) = s_i - \frac{1}{2n}(\sum_{j=1}^n s_j)^2 - rs_i$, $S_i = [0,\bar{s}]$, $R = [0,\bar{r}]$, and true feedback $f(s) = \alpha \sum_{j=1}^n s_j$, $\alpha > 0$. Players perceive $r = \hat{f}(s) = \hat{\alpha} \sum_j s_j$, where $\hat{\alpha}$ is their belief, initially $\hat{\alpha}_0$, unknown to equal α. Over discrete periods $t = 0, 1, \ldots,$ players choose s^t, observe $r^t = f(s^t)$, and update $\hat{\alpha}_{t+1}$ via Bayesian learning.

E.2 Learning Dynamics

At t, given $\hat{\alpha}_t$, players compute an SFE assuming $r = \hat{\alpha}_t \sum_j s_j$. FOC:

$$1 - s - r = 0, \quad r = \hat{\alpha}_t ns, \tag{24}$$

$$s = 1 - \hat{\alpha}_t ns \quad \Rightarrow \quad s^t = \frac{1}{1+\hat{\alpha}_t n}, \quad r^t = \frac{\hat{\alpha}_t n}{1+\hat{\alpha}_t n}. \tag{25}$$

But $r^t = f(s^t) = \alpha n s^t = \frac{\alpha n}{1+\hat{\alpha}_t n}$. Assume $r^t = \alpha n s^t + \epsilon_t$, $\epsilon_t \sim N(0,\sigma^2)$ (noise from measurement or shocks). Prior $\hat{\alpha}_0 \sim N(\mu_0, \tau_0^2)$. Posterior $\hat{\alpha}_{t+1}$ given r^t updates via Bayes' rule:

$$r^t = \alpha n s^t + \epsilon_t, \quad s^t = \frac{1}{1+\hat{\alpha}_t n}, \tag{26}$$

likelihood $r^t \sim N(\alpha n s^t, \sigma^2)$. Posterior mean:

$$\hat{\alpha}_{t+1} = \frac{\tau_0^{-2}\mu_0 + (ns^t/\sigma^2) r^t}{\tau_0^{-2} + (ns^t)^2/\sigma^2}, \tag{27}$$

variance $\tau_{t+1}^2 = \left(\tau_0^{-2} + \frac{(ns^t)^2}{\sigma^2}\right)^{-1}$. As $t \to \infty$, $\hat{\alpha}_t \to \alpha$ (consistency), and $s^t \to s^* = \frac{1}{1+\alpha n}$.

E.3 SFE with Learning

A learning-augmented SFE is a sequence $\{(s^{t*}, r^{t*}, \hat{\alpha}_t)\}$ where s^{t*} is Nash given $\hat{\alpha}_t$, $r^{t*} = f(s^{t*})$, and $\hat{\alpha}_{t+1}$ updates. Existence at each t follows Section 2.1 (compact S, continuous u_i). Convergence requires simulation, but $\hat{\alpha}_t$ stabilizes at α, recovering the true SFE.

E.4 Discussion

Learning delays lock-in if $\hat{\alpha}_0 < \alpha$ (underestimated feedback), aligning with adaptive behavior (Fudenberg and Levine, 1998). Future work could model strategic manipulation of f's perception or continuous-time learning.